Convexity Adjustments for Affine Term Structure Models

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Motivation

- huge market for interest rate derivatives (around \$500 tln at the end of 2011, according to BIS)
- many exotic interest rate products: mismatch in time between the interest rate ($y(T)$) and the time of payment ($S \neq T$)
- examples:
  - *LIBOR in arrears*: one observes $L(T, S)$ and receives the payment at the same time, T, instead of waiting one period and receiving payment at S.
  - *Constant Maturity Swap*: the floating leg of the swap pays the swap rate with maturity c years.

This talk

- Brief introduction to convexity correction
- **ONE** set up for various types of convexity corrections
- Framework consistent with no arbitrage and employing known tools from affine term structure theory
- **EXACT** solutions up to an ODE
- Examples

Various “Convexities”

- Yield Convexity Adjustments
- Forward versus futures price adjustments
- **Modified schedule or timing adjustments**
  - LIBOR in arrears adjustments
  - Constant Maturity Swap adjustments
  - ...
- Mismatch between the domestic currency of an underlying asset and the settlement payment of the derivative claim or quanto adjustments.

Connection to the Radon-Nykodim Derivative

rate $y(T)$ with time to maturity T paid at S

$$V(t) = E_t^Q[e^{-\int_t^S r(u)du}\Phi(y(T))] = p(t,S)E_t^S[\Phi(y(T))].$$

- Q^S - EMM with $p(t,S)$ numeraire
- Q^Φ - EMM where $\Phi(y(T))$ is a martingale
  - $y(T)$ is a LIBOR rate, $Q^\Phi = Q^T$
  - $y(T)$ is a swap rate, Q^Φ is the appropriate swap measure
- change of measure given by $\frac{dQ^\Phi}{dQ^S} = \Lambda_T$ on $\mathcal{F}_T$

$$\begin{aligned} E_t^S[\Phi(y(T))] &= E_t^\Phi[\Phi(y(T))\Lambda_T] \\ &= \Phi(y(t))\Lambda_t + cov[\Phi(y(T)),\Lambda_T]. \end{aligned}$$

due to Antoon Pelsser

How are the products evaluated?

- first, compute the price as if our payoff would have "nice" martingale properties
- adjust that price for what is known as *convexity adjustments*.

$$E_t^S[\Phi(y(T))] = \Phi(y(t)) + \underbrace{\mathbf{C}^\Phi(t, T, S)}_{\text{convexity correction}} \quad .$$

versus $\Phi(y(t))\Lambda_t + cov[\Phi(y(T)), \Lambda_T]$

in general, convexity corrections are

- model dependent
- require numerical approximation methods for the joint distribution of $\Phi(y(T))$ and $p(t, S)$

Traditional Approach

- Write our payoff as a function G of Z_t that we know it is a martingale under the $\mathcal{Q}^S$ measure.

$$\Phi_t = G(Z_t)$$

- Use second order Taylor approximations to obtain.

$$\Phi_T \approx G(Z_t) + G'(Z_t)(Z_T - Z_t) + \frac{1}{2}G''(Z_t)(Z_T - Z_t)^2$$

- Take expectations on both sides

$$E_t^S[\Phi_T] \approx \underbrace{G(Z_t)}_{\Phi_t} + G'(Z_t)\underbrace{\left(E_t^S[Z_T] - Z_t\right)}_{0} + \underbrace{\frac{1}{2}G''(Z_t)E_t^S\left[(Z_T - Z_t)^2\right]}_{\approx CC^{\Phi}(t)}$$

Up to now...

Practitioners

- Use various (*ad hoc*) rules to calculate convexity adjustments for different products.

Previous Literature

- Proposed formulas:
  - Based upon different interest rate modeling approaches, depending on the products.
    (e.g. formulas from LIBOR market models to products that depend on LIBOR rates, Swap market models to products that depend on swap rates, etc...)
  - Most of the time, based upon Gaussian assumptions
  - if you want to avoid Gaussian assumptions, you need to rely on Taylor approximations.

However...

we have the tools necessary
to do better.
(in some cases)

Affine Term Structures

under the risk-neutral measure Q

- factors $dZ_t = \alpha(t, Z_t)dt + \sigma(t, Z_t)dW_t$
  with $\alpha(t, Z_t) = d(t) + E(t)Z_t$
  and $\Sigma = \sigma(t, Z_t)\sigma^\top(t, Z_t) = k_0(t) + \sum_{i=1}^m k_i(t)Z_t^i$
- bond prices $p(t, T) = \exp\left\{A(t, T) + B^\top(t, T)Z_t\right\}$

$$dp(t, T) = rp(t, T)dt + \underbrace{B^{\top(t,T)}\sigma(t, Z_t)}_{v(t,T,Z)}dW_t$$

Question

Do the Affine Term Structure models allow us to compute convexity corrections in a better and more consistent way?

- for LIBOR based exotic products
- for swap products

LIBOR-in-arrears

- quoted LIBOR $L(T,S)$
- modelling the forward LIBOR $L(t,T,S)$
- one observes $L(T,S)$ and receives the payment at the same time, T, instead of waiting one period and receiving payment at S.

$$p(t,T)E^T[L(T,S)] = \frac{p(t,S)}{S-T}E^S\left[\frac{p(T,T)}{p(T,S)}\right] - \frac{p(t,T)}{S-T}$$

LIA adjustment

Ansatz for $T \leq U \leq S$

$$\underbrace{E_t^U\left[\frac{p(T,T)}{p(T,S)}\right]}_{M_t} = \frac{p(t,T)}{p(t,S)} e^{[F(t,T,U,S)+G^\top(t,T,U,S)Z_t]}$$

- M_t, $E_t^U\left[\frac{p(T,T)}{p(T,S)}\right]$, i.e. is a martingale under Q^U
- compute dynamics of $\frac{p(t,T)}{p(t,S)}$ and Z_t under Q^U
- apply Itô to the right hand side

$$\frac{p(t,T)}{p(t,S)} e^{[F(t,T,U,S)+G^\top(t,T,U,S)Z_t]}$$

  to compute dynamics of M_t and set the drift to zero.
- separate the free terms and the Z_t- terms

Note: In ATS, var-cov matrices are AFFINE!

Ricatti ODE's

free terms

$$
\begin{aligned}
&F_t + [B(t,S) - B(t,U)]^\top k_0(t)[B(t,T) - B(t,S)] + \\
&+[d(t) - B^\top(t,U)k_0(t)]G + \frac{1}{2}G^\top k_0(t)G + \\
&+[B(t,T) - B(t,S)]^\top k_0(t)G = 0 \\
&F(T,T,S) = 0
\end{aligned}
$$

the z-terms

$$
\begin{aligned}
&G_t + [\bar{B}(t,S) - \bar{B}(t,U)]^\top K[B(t,T) - B(t,S)] + \\
&+[E(t)z - \bar{B}^\top(t,U)K]G + \frac{1}{2}G^\top KG + \\
&+[\bar{B}(t,T) - \bar{B}(t,S)]^\top KG = 0 \\
&G(T,T,S) = 0
\end{aligned}
$$

Why do things work out?

Note!

$$
\begin{aligned}
& F(t,T,U,S) + G^\top(t,T,U,S)Z_t \\
= \; & \underbrace{\int_t^T [v(s,T) - v(s,S)][v(s,U) - v(s,S)]^\top ds}_{\text{var-cov matrix for } \frac{p(t,T)}{p(t,S)} \text{ and } \frac{p(t,U)}{p(t,S)}}
\end{aligned}
$$

we obtain

$$
C_{LIA} = \frac{p(t,T)}{S-T}\left[\frac{p(t,T)}{p(t,S)}e^{F(t,T,U,S)+G(t,T,U,S)Z_t} - 1\right]
$$

Example: LIA CC CIR Model

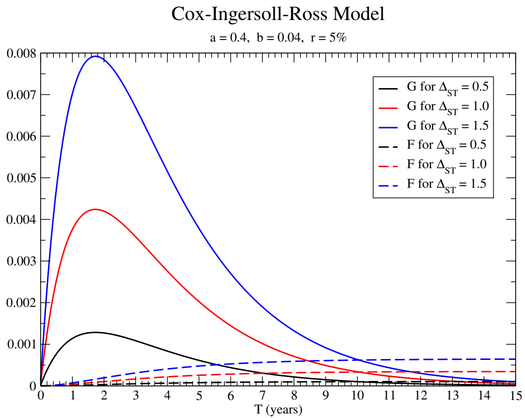
$$dr = a(b-r)dt + \sigma\sqrt{r}dW$$

- Take $U = T$
- Take $S = T + \Delta_{ST}$

$$CC^{LIA}(t,T,T+\Delta_{ST}) = \frac{1}{\Delta_{ST}} \frac{p(t,T)}{p(t,T+\Delta_{ST})} \times \times \left[e^{F(t,T,T+\Delta_{ST}) + G(t,T,T+\Delta_{ST})^{\top} Z_t} - 1 \right]$$

$$\begin{cases} \frac{\partial F}{\partial t} + a\,b\,G = 0\,, \\ \frac{\partial G}{\partial t} + \sigma^2[B(t,S)-B(t,T)][B(t,T)-B(t,S)] - \left[\sigma^2 B(t,S) + a\right] G + \frac{\sigma^2}{2} G^2 = 0\,. \end{cases}$$

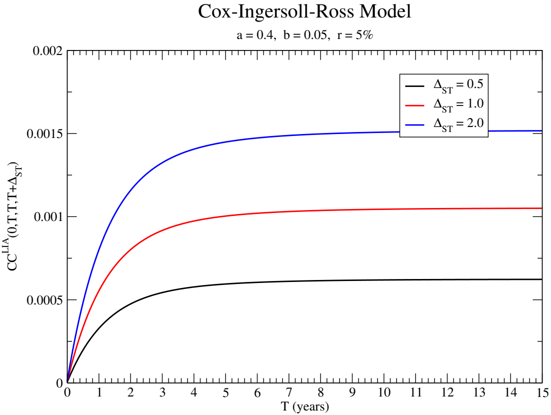
Example: LIA CC CIR Model



Typical $F(0, T, T + \Delta_{ST})$ and $G(0, T, T + \Delta_{ST})$ for different Δ_{ST} .

Gaminha, Gaspar & Oliveira (2012)

Example: LIA CC CIR Model



$CC^{LIA}(0, T, T + \Delta_{ST})$ for the various Δ_{ST}.

Gaminha, Gaspar & Oliveira (2012)

CMS

- **Constant Maturity Swap** refers to a swap rates which fixes in the future and have **constant** time-to-maturity c.
- A fixed for floating CMS swap is a periodic exchange of interest payments on a fixed notional in which the floating rate is indexed by a reference swap rate (say, the 10 year swap rate)
- At time T_i the **floating leg** pays

$$S(T_{i-1}, T_i, T_{i+c})$$

- CMS rates provide a convenient alternative to LIBOR as a floating index.

Swap Rates in ATS

Notation:

- $S(t, T_n, T_N) = \frac{p(t,T_0)-p(t,T_N)}{P(t,T_0,T_N)}$ - forward swap rate
- $P(t, T_n, T_N) = \sum_{i=1}^{N} \alpha_i p(t, T_i)$
- Q^{S_N} - EMM with $P(t, T_n, T_N)$ as numeraire;

Schrager& Pelsser(2006)

- **affine approximate dynamics** for $S(t, T_n, T_N)$

  Assumption: $\frac{p(t,T_j)}{P(t,T_n,T_N)}$ are low variance martingales

$$dS(t, T_n, T_N) = \left[\sum_{i=n}^{N} q_j(0) B(t, T_j)^\top\right] \sigma(t, Z_t) dW_t^{S_N}$$

CMS Convexity Adjustment

CMS rate

$$K = \frac{\sum_{i=1}^{n} \alpha_i P(0, T_i) E_t^{T_i} S(T_{i-1}, T_i, T_{i+c})}{\sum_{i=1}^{n} \alpha_i P(0, T_{i-1})}$$

key component

$$\overbrace{E_t^{T_i}[S(T_{i-1}, T_i, T_{i+c})]}^{M_t} \\ = S(t, T_i, T_{i+c}) + F(t, T_i, T_{i+c}) + G^\top(t, T_i, T_{i+c}) Z_t$$

we proceed as before...

- M_t,i.e. $E_t^{T_i}[S(T_{i-1}, T_i, T_{i+c})]$, is a martingale under Q^{T_i}
- compute dynamics of $S(t, T_i, T_{i+c})$ and Z_t, the factors, under Q^{T_i}
- apply Itô to the right hand side,

$$S(t, T_i, T_{i+c}) + F(t, T_i, T_{i+c}) + G^\top(t, T_i, T_{i+c})Z_t,$$

  to compute dynamics of M_t and set the drift to zero.
- separate the free terms and the Z_t- terms
- obtain a set of Ricatti ODE's

Ricatti ODE's

$$F_t - \left[\sum_{j=i}^{i+c} q_j(0) B(t, T_j)^*\right] k_0(t) B(t, T_i)$$
$$+[d(t) - B^*(t, T_i) k_0(t)] G = 0$$
$$F(T_i, T_i) = 0$$

and

$$G_t - \left[\sum_{j=i}^{i+c} q_j(0) \bar{B}(t, T_j)^*\right] K B(t, T_i)$$
$$+[E(t) z - \bar{B}(t, T_i) K] G(t, T, S) = 0$$
$$G(T_i, T_i) = 0$$

Practical implication

- we take advantage of the structure imposed by ATS to develop **one** framework for computing convexity corrections for timing adjustments
- we can obtain **close form solutions(up to an ODE)** for LIA and CMS in a non-Gaussian setting
- these rates are going to be consistent with no arbitrage principles
- since the rates are modeled in an ATS setting it is fairly straight forward to compute prices for caplets and florlets, a.s.o.

THANK YOU!