Arbitrage free modelling of the bond market

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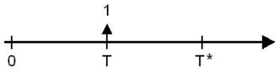
Outline

Bond market - introduction

A **bond** is a financial contract paying 1 EUR at date $T > 0$.



T - maturity of the bond; known at time 0;

$P(t, T)$ - the value of the bond with maturity T at time t;

- $P(\cdot, T)$ - some stochastic process on $[0, T]$ with $P(T, T) = 1$; $\quad T > 0$,
- $P(t, \cdot)$ - a function describing the bond prices at time t;

Savings account:

$r(\cdot)$ - a short rate process,

The value at time t of 1 EUR paid at time T:

$$e^{-\int_t^T r(s)ds} \longleftarrow 1 \text{ at time } T;$$

Model of the bond prices:

$$P(t,T) := e^{-\int_t^T f(t,u)du},$$

$f(\cdot,\cdot) = f(\omega,\cdot,\cdot)$ - forward rate,

Difference: For $t < T$

- $r(T)$ is not known at time t;
- $f(t,T)$ is known at time t.

Forward rate model

The forward rate f:

- fully determines the bond prices,
- determines the short rate process by: $r(t) := f(t,t)$,
- specifies other random quantities on the market, for instance the semiannual *LIBOR* rate:

$$1 + 0,5 \cdot LIBOR(t,x) = e^{\int_x^{x+0,5} f(t,t+u)du}.$$

f can be treated as

- $f(\cdot, T)$ a stochastic process
  $\longrightarrow$ a traditional approach,
- $f(t, \cdot)$ as an element of some Hilbert space
  $\longrightarrow$ the *SPDE* approach,
- $(t,T) \longrightarrow f(t,T)$ as a random function
  $\longrightarrow$ a random field approach.

The forward rate dynamics

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dL(t), \qquad 0 < t \leq T < +\infty,$$
$$f(0,T) = f_0(T)$$

where L is a general Lévy process. For L - Wiener process, see ($\diamondsuit$).

Question: When the market is arbitrage-free? When the discounted bond prices

$$\hat{P}(t,T) = e^{-\int_0^T f(t,u)du}, \qquad 0 < t \leq T$$

are local martingales?

($\diamondsuit$) Heath, D., Jarrow, R., Morton, A.: Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation, Econometrica **60**, 77-105 (1992)

Heath-Jarrow-Morton condition (HJM)

The bond market is arbitrage-free if and only if

$$(\diamondsuit) \quad \int_t^T \alpha(t,u)du = J\left(\int_t^T \sigma(t,u)du\right), \qquad (HJM - condition)$$

where J is a Laplace exponent of L:

$$\mathbf{E}(e^{-zL(t)}) = e^{tJ(z)}, \qquad t \geq 0,\ z \in \mathbb{R}.$$

By taking the T-derivatives we obtain:

$$\alpha(t,T) = J'\left(\int_t^T \sigma(t,u)du\right)\sigma(t,T).$$

($\diamondsuit$) Eberlein, E., Raible, S.:"Term structure models driven by general Lévy processes", (1999), *Math. Finance*, 9, 31-53.

($\diamondsuit$) Jakubowski, J., J. Zabczyk : "Exponential moments for HJM models with jumps", (2007), *Finance and Stochastics*, 11, 429-445.

Forward rate dynamics:

$$df(t,T) = J'\left(\int_t^T \sigma(t,u)du\right)\sigma(t,T)dt + \sigma(t,T)dL(t) \;\; (*)$$

$$f(0,T) = f_0(T).$$

Remark: The bond market model is specified by

- $J'(\cdot)$,
- $\sigma(\cdot,\cdot)$.

If $\sigma = \sigma(f)$ then $(*)$ becomes an equation.

Problem: Does the HJM equation have a solution ?

HJMM equation

Remark: $f(t,\cdot)$ have domains $[t,+\infty)$ - dependent on time!

Musiela paramterization $\longrightarrow$ use $x := T - t$ **(time to maturity)** instead of T ;

$$r(t,x) := f(t,t+x), \quad x \geq 0,$$

$$F(t,x) := \alpha(t,t+x), \quad G(t,x) := \sigma(t,t+x),$$

$$r(0,x) := r_0(x) = f_0(x) = f(0,x), \quad x \geq 0.$$

Then all functions $r(t,\cdot)$ have the same domain $[0,+\infty) \longrightarrow r$ may be treated as a Hilbert space valued process.

The dynamics of r:

$$r(t,x) = f(t,t+x) = f(0,t+x) + \int_0^t \alpha(s,t+x)ds + \int_0^t \sigma(s,t+x)dL_s$$

$$= r(0,x) + \int_0^t F(s,t-s+x)ds + \int_0^t G(s,t-s+x)dL_s.$$

$\longrightarrow r$ is a weak solution of the SPDE;

Then the (HJM) equation transforms to the (HJMM) equation:

$$f(t,T) = f_0(T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)dL(s) \qquad (\textit{HJM equation})$$

$$\Updownarrow$$

$$dr(t,x) = \left[\frac{\partial r}{\partial x}(t,x) + F(t,x)\right] dt + G(t,x)dL(t) \qquad (\textit{HJMM equation})$$

Weak solution of the HJMM equation

$$r(t,x) = S_t r_0(x) + \int_0^t [S_{t-s}F(s,x)]\,ds + \int_0^t S_{t-s}G(s,x)]dL(s)$$

Shift semigroup:

$$S_t h(x) = h(t+x), \qquad t \geq 0,\ x \geq 0.$$

Conclusion: The HJMM equation is an SPDE of the form

$$dr(t,x) = (Ar(t,x) + F(r(t))(x))dt + G(r(t))(x)dL(t)$$

where

$$Ah(x) = \frac{d}{dx}h(x),$$

$$F(h)(x) = J'\Big(\int_0^x G(h)(v)dv\Big)G(h)(x)$$

State spaces

- $L^{2,\gamma}$

$$\|h\|^2_{L^{2,\gamma}} := \int_0^{+\infty} \mid h(x) \mid^2 e^{\gamma x}dx < +\infty,$$

- $H^{1,\gamma}$

$$\|h\|^2_{H^{1,\gamma}} := \int_0^{+\infty} \Big(\mid h(x) \mid^2 + \mid h'(x) \mid^2\Big) e^{\gamma x}dx < +\infty,$$

where $\gamma > 0$.

The HJMM equation is specified by J' and $G : H \longrightarrow H$ where

$$G(h)(x) = g(x, h(x)), \quad \text{GENERAL CASE}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.

$$g(x, r) = \lambda(x)r, \quad \text{LINEAR CASE},$$

$\lambda(\cdot)$ deterministic function.

Problem: Does the HJMM equation have solutions?

Aim: Formulate conditions on g and J' such that

- there exists solution to the HJMM eq.
- the solution is positive.

SPDE - existence and positivity

EXISTENCE

(♢) **Theorem (Peszat, Zabczyk)**

$$dX = (AX + F(X))\,dt + G(X-)dL(t)$$

- *linear growth*

$$\| F(x) \|_H + \| G(x) \|_H \leq c(1 + \| x \|_H)$$

- *Lipschitz condition*

$$\| F(x) - F(y) \|_H + \| G(x) - G(y) \|_H \leq c(\| x - y \|_H)$$

- *local Lipschitz condition*

$$\forall R > 0\ \exists c_R > 0 \text{ such that } \forall x, y \in H, \| x \|_H, \| y \|_H \leq R$$

$$\| F(x) - F(y) \|_H + \| G(x) - G(y) \|_H \leq c_R(\| x - y \|_H)$$

Then

- *linear growth + Lipschitz condition* $\Longrightarrow$ $\exists!$ *weak solution;*
- *linear growth + local Lipschitz condition* $\Longrightarrow$ $\exists!$ *weak solution.*

(♢) Peszat, Sz., Zabczyk J.: "Stochastic partial differential equations with Lévy noise",(2007), Cambridge University Press

POSITIVITY

($\diamondsuit$) **Theorem [Milian]**

- *semigroup S_t preserves positivity,*
- *F, G - Lipschitz transformations*

Then the equation

$$dX = (AX + F(X))\, dt + G(X) dW(t),$$

$$W - \textit{real Wiener process}, \ H = L^2.$$

preserves positivity if and only if

$$\forall \varphi \geq 0 \quad \textit{and} \quad \phi \geq 0 \quad s.t. \langle \varphi, \phi \rangle = 0 \quad \Longrightarrow \quad \langle F(\varphi), \phi \rangle \geq 0, \ \langle G(\varphi), \phi \rangle = 0.$$

Generalization

- L - Lévy process,
- F, G - **locally** Lipschitz transformations,

$\longrightarrow$ Consider L without small jumps and formulate conditions on positivity at the jump moments. Then pass to the limit.

($\diamondsuit$) Milian, A. : "Comparison theorems for stochastic evolution equation", (2002), *Stochastics and Stochastic Reports*, 72, 79-108.

General HJMM equation

Theorem

Assume that F, G are locally Lipschitz in $L^{2,\gamma}$.

i) *Then the HJMM equation is positivity preserving if and only if*

$$\begin{aligned} r + g(x,r)u &\geq 0 \qquad && r \geq 0,\ x \geq 0,\ u \in supp\,\nu, \\ g(x,0) &= 0 && x \geq 0 \end{aligned} \tag{1}$$

ii) *If $supp\{\nu\} \subseteq [-m, +\infty)$ and $g \geq 0$ then (1) holds iff $g(x,r) \leq r/m$.*

iii) *If $g(x,r) = \lambda(x)r$ where $\bar{\lambda} := \sup_{x \geq 0} \lambda(x)$ then (1) holds iff*

$$supp\{\nu\} \subseteq [-\frac{1}{\bar{\lambda}}, +\infty). \tag{2}$$

• Local existence

$$(G1) \begin{cases} (i) & \text{The function g is continuous on } \mathbb{R}^2_+ \text{ and} \\ & g(x,0)=0,\ g(x,y)\geq 0, \quad x,y\geq 0. \\ (ii) & \text{For all } x,y\geq 0 \text{ and } u\in \operatorname{supp}\nu : \\ & x+g(x,y)u\geq 0. \\ (iii) & \text{There exists a constant } C>0 \text{ such that} \\ & \mid g(x,u)-g(x,v)\mid \leq C\mid u-v\mid, \qquad x,u,v\geq 0. \end{cases}$$

Theorem [loc. ex. in $L^{2,\gamma}_+$]

Assume that $(G1)$ holds and either L is a Wiener process or for some $z_0 > 0$:

$$(*) \quad \int_{-\infty}^{-1} \mid y \mid^2 e^{z_0 |y|}\nu(dy) < +\infty, \quad \textit{and} \quad \int_1^{+\infty} y^2\nu(dy) < +\infty.$$

Then for arbitrary initial condition $r_0 \in L^{2,\gamma}_+$ there exists a unique local solution to the HJMM eq. in $L^{2,\gamma}_+$.

$(*) \iff J'$ is locally Lipschitz

$$
(G2)\quad \begin{cases} (i) & \text{The functions } g'_x, g'_y \text{ are continuous on } \mathbb{R}^2_+ \text{ and} \\ & \quad g'_x(x,0) = 0, \qquad x \geq 0. \\ (ii) & \sup_{x,y\geq 0} \mid g'_y(x,y) \mid < +\infty, \\ (iii) & \text{There exists a constant } C > 0 \text{ such that for,} \qquad x,u,v \geq 0 \\ & \quad \mid g'_x(x,u) - g'_x(x,v) \mid + \mid g'_y(x,u) - g'_y(x,v) \mid \leq C \mid u - v \mid . \end{cases}
$$

Theorem [loc. ex. in $H^{1,\gamma}_+$]

Assume that $(G1)$ and $(G2)$ hold and for some $z_0 > 0$

$$
(*) \qquad \int_{-\infty}^{-1} \mid y \mid^3 e^{z_0 \mid y \mid} \nu(dy) < +\infty, \quad \textit{and} \quad \int_1^{+\infty} y^3 \nu(dy) < +\infty.
$$

Then for arbitrary initial condition $r_0 \in H^{1,\gamma}_+$ there exists a unique local solution to HJMM eq. in $H^{1,\gamma}_+$.

$(*) \iff J', J''$ are locally Lipschitz

Conclusion: If

- L is a Wiener process or
- L has small jumps only

then there exists a local solution.

• Global existence

Theorem [glob. ex. in $L^{2,\gamma}_+$]

Assume that $(G1)$ *holds and in addition:*

$$(*) \quad q=0, \quad supp\{\nu\} \subseteq [0,+\infty), \quad \int_0^{+\infty} \max\{y, y^2\}\nu(dy) < +\infty.$$

Then for arbitrary $r_0 \in L^{2,\gamma}_+$ *the HJMM eq. has unique global solution in* $L^{2,\gamma}_+$.

$(*) \iff J'$ is locally Lipschitz and bounded on $[0,+\infty)$

$$(G3) \quad \begin{cases} (i) & \text{Partial derivatives } g'_y, g''_{xy}, g''_{yy} \text{ are bounded on } \mathbb{R}^2_+. \\ (ii) & 0 \leq g(x,y) \leq c\sqrt{y}, \qquad x, y \geq 0, \\ (iii) & \mid g'_x(x,y) \mid \leq h(x), \qquad x, y \geq 0, \ \text{for some } h \in L^{2,\gamma}_+. \end{cases}$$

Theorem [gl. ex. in $H^{1,\gamma}_+$]

Assume that conditions $(G1)$, $(G2)$ and $(G3)$ are satisfied and

$$(*) \quad q = 0, \quad supp\{\nu\} \subseteq [0, +\infty), \quad \int_0^{+\infty} \max\{y, y^3\}\nu(dy) < +\infty.$$

Then for arbitrary $r_0 \in H^{1,\gamma}_+$ there exists a unique global solution to the HJMM eq. in $H^{1,\gamma}_+$.

$(*) \iff J', J''$ are locally Lipschitz and bounded on $[0, +\infty)$

Conclusion: The standard SPDE methods exclude from the analysis all Lévy processes which

- have Wiener part,
- have negative jumps.

Linear HJMM equation

Linear case

$$g(x,r) = \lambda(x) r, \quad x, r \geq 0,$$

$\lambda(\cdot)$ - continuous function,

$$\underline{\lambda} := \inf_{x \geq 0} \lambda(x), \qquad \bar{\lambda} := \sup_{x \geq 0} \lambda(x).$$

Assumptions on positivity reduce to:

$$(\Lambda 0) \quad \begin{cases} (i) \quad \underline{\lambda} > 0, \\ (ii) \quad \operatorname{supp} \nu \subseteq [-\frac{1}{\bar{\lambda}}, +\infty), \\ (iii) \quad \bar{\lambda} < +\infty. \end{cases}$$

• Local existence

From the general case the following results can be deduced.

Theorem [loc. ex. in $L^{2,\gamma}_{+}$]
Assume that $(\Lambda 0)$ *and*

$$\int_1^{+\infty} y^2 \nu(dy) < +\infty,$$

hold. Then there exists a unique local weak solution to the HJMM eq. taking values in the space $L^{2,\gamma}_{+}$.

Theorem [loc. ex. in $H^{1,\gamma}_{+}$]
Assume that conditions $(\Lambda 0)$,

$$\lambda' \text{ is bounded and continuous on } \mathbb{R}_+,$$

and

$$\int_1^{+\infty} y^3 \nu(dy) < +\infty,$$

are satisfied. Then there exists a unique local weak solution to the HJMM eq. taking values in the space $H^{1,\gamma}_{+}$.

• Global existence

Remark: Linear case is not captured by the general case
$\longrightarrow$ F does not satisfy linear growth condition.

Proposition

If F satisfies the linear growth condition in $L^{2,\gamma}$ then J' is bounded on $[0,+\infty)$.

Remark: J' is bounded $\Longleftrightarrow$ L is a subordinator;

$\longrightarrow$ the standard SPDE methods capture only models driven by the subordinators;

Forward rates in standard parametrization

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We treat the forward rate f as a **bounded random field on a bounded domain** $\mathcal{T}$:

$$\mathcal{T} := \left\{(t,T) \in \mathbb{R}^2 : 0 \leq t \leq T \leq T^*\right\}.$$

which satisfies

- $f(\cdot,T)$ is adapted and càdlàg on $[0,T]$ for all $T \in [0,T^*]$,
- $f(t,\cdot)$ is continuous on $[t,T^*]$ for all $t \in [0,T^*]$,
- $P(\sup_{(t,T)\in\mathcal{T}} f(t,T) < \infty) = 1$.

The growth conditions for J':

- $\limsup_{z\to\infty} \left(\ln z - \bar{\lambda} T^* J'\left(z\right)\right) = +\infty, \qquad 0 < T^* < +\infty, \qquad (L1)$
- $J'(z) \geq a(\ln z)^3 + b, \qquad \forall z > 0, \text{ for some } a > 0, b \in \mathbb{R} \qquad (L2).$

Theorem ($\diamondsuit$)

a) $(L1) \implies$ there exists a bounded field solving the linear equation,
b) $(L2) \implies$ there is no bounded field solving the linear equation.

($\diamondsuit$) Barski M., Zabczyk J.: "Forward rate models with linear volatilities", (2012) *Finance and Stochastics* 16, 3, p. 537-560.

Alternative method for the linear HJMM equation

Operator equation:

$$r(t,x) = \mathcal{K}(r)(t,x), \quad r(\cdot,\cdot) \text{ - random field,}$$

where

$$\mathcal{K}h(t,x) = a(t,x)e^{\int_0^t J'\left(\int_0^{t-s+x} \lambda(v)h(s,v)dv\right)\lambda(t-s+x)ds}, \quad x \geq 0, \quad t \in (0, T^*].$$

and

$$a(t,x) := r_0(t+x)e^{\int_0^t \lambda(t-s+x)dL(s) - \frac{q^2}{2}\int_0^t \lambda^2(t-s+x)ds}$$

$$\cdot \prod_{0 \leq s \leq t} \left(1 + \lambda(t-s+x)\triangle L(s)\right) e^{-\lambda(t-s+x)\triangle L(s)}.$$

Proposition

Under some mild assumptions:

- *r takes values in $L^{2,\gamma}$ and solves operator equation $\Longrightarrow$ r solves HJMM in $L^{2,\gamma}$,*
- *r takes values in $H^{1,\gamma}$ and solves operator equation $\Longleftrightarrow$ r solves HJMM in $H^{1,\gamma}$.*

For the proof we need regularity of the random fields

$$I_1(t,x) := \int_0^t \lambda(t-s+x)dL(s), \quad t \in [0,T^*], \ x \geq 0,$$

$$I_2(t,x) := \prod_{0\leq s\leq t} (1+\lambda(t-s+x)\triangle L(s))\, e^{-\lambda(t-s+x)\triangle L(s)}, \qquad t\in[0,T^*], x\geq 0,$$

Proposition

- If $\lambda(\cdot)$ is bounded and continuous then $I_1(\cdot,\cdot)$ is bounded and $I_1(\cdot,x)$ is càdlàg.
- If $\lambda(\cdot), \lambda'(\cdot)$ are bounded and continuous then $I_2(\cdot,\cdot)$ is bounded and $I_2(\cdot,x)$ is càdlàg.

Growth conditions for J':

- $\limsup_{z\to\infty} \left(\ln z - \bar{\lambda} T^* J'\left(z\right)\right) = +\infty, \qquad 0 < T^* < +\infty, \qquad (L1)$
- $J'(z) \geq a(\ln z)^3 + b, \qquad \forall z > 0$, for some $a > 0, b \in \mathbb{R} \qquad (L2)$.

Theorem [non-existence in $H_+^{1,\gamma}$]

Assume that conditions $(\Lambda 0)$ *,*

- $\lambda, \lambda', \lambda''$, *are bounded and continuous on* $\mathbb{R}_+$*,*
- $\int_1^{+\infty} y\nu(dy) < +\infty$ *,*
- $(L2)$

are satisfied.
Then, for some $k > 0$ *and all* $r_0(\cdot) \in H_+^{1,\gamma}$ *such that* $r_0(x) \geq k, \forall x \in [0, T^*]$*, the global solution in* $H_+^{1,\gamma}$ *of HJMM eq. does not exist on the interval* $[0, T^*]$*.*

Corollary: Each local solution in $H_+^{1,\gamma}$ explodes under $(L2)$.

Theorem [glob. ex.]

Assume that $(\Lambda 0)$ *and conditions*

- λ, λ' *are bounded and continuous on* $\mathbb{R}_+$,
- $\int_1^{+\infty} y\nu(dy) < +\infty$,
- $(L1)$ $\limsup_{z\to\infty} \left(\ln z - \bar{\lambda} T^* J'(z)\right) = +\infty, \qquad 0 < T^* < +\infty,$

hold.

(a) If $r_0 \in L^{2,\gamma}_+$ *then there exists a solution to the HJMM eq. taking values in the space* $L^{2,\gamma}_+$.

(b) Assume, in addition, that

- λ'' *is bounded and continuous on* $\mathbb{R}_+$,
- *supp*$\{\nu\} \subseteq [0,+\infty)$ *and* $\int_1^{\infty} y^2\nu(dy) < \infty$.

If $r_0 \in H^{1,\gamma}_+$ *then there exists a solution to the HJMM eq. taking values in the space* $H^{1,\gamma}_+$.

Theorem [UNIQUENESS]
Assume that

$$supp\{\nu\} \subseteq (0,+\infty) \quad and \quad \int_1^\infty y^2 \nu(dy) < \infty,$$

is satisfied. If, on the interval $[0, T^*]$*, there exists a non-exploding solution of the HJMM eq. taking values in* $L^{2,\gamma}_+$ *then it is unique.*

Comment: Our assumptions for existence are weak. They do not even imply local Lipschitz conditions.

Corollary:

- L has the Wiener part $\rightarrow$ no solutions,
- L has negative jumps $\rightarrow$ no solutions.
- L is a subordinator with drift $\rightarrow$ there are solutions.

• Characteristics of the noise and existence

In general the answer depends on the behavior of the function

$$U_\nu(x) := \int_0^x y^2 \nu(dy), \qquad x \geq 0,$$

near the origin.

A positive function L *varies slowly at* 0 if for any fixed $x > 0$

$$\frac{L(tx)}{L(t)} \longrightarrow 1, \qquad \text{as } t \longrightarrow 0.$$

Typical examples:

$$L(t) \equiv const., \quad L(t) = \left(\ln \frac{1}{t}\right)^\gamma, \gamma > 0 \text{ for small positive } t.$$

Theorem

Assume that for some $\rho \in (0, +\infty)$,

$$U_\nu(x) \sim x^\rho \cdot L(x), \qquad \text{as } x \to 0,$$

where L is a slowly varying function at 0*.*

i) *If* $\rho > 1$ *then there exists a solution.*

ii) *If* $\rho < 1$*, then there is no solution.*

iii) *If* $\rho = 1$*, the measure* ν *has a density and*

$$L(x) \longrightarrow 0 \quad \text{as } x \to 0, \quad \text{and} \quad \int_0^1 \frac{L(x)}{x}\, dx = +\infty,$$

then there exists a solution.

THANK YOU FOR ATTENTION :-)