Malliavin calculus method for asymptotic expansion of dual control problems

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Outline

Goal of talk

- Develop an approximation scheme for computing the value function of entropy-weighted stochastic control problems.
- We do this by treating the control as a small perturbation to the path of the state variable on Wiener space. This uses ideas related to the stochastic calculus of variations.
- We do not require the state process to be Markovian, and the terminal payoff is a general (hence path-dependent) functional of the paths of the state variable.

Entropy weighted stochastic control problem

- The type of stochastic control problem we analyse typically arises in the dual approach to exponential indifference valuation of claims.
- The problem is an optimisation over equivalent local martingale measures (ELMMs) $\mathbb{Q}$ and maximises an expectation of a random payoff, penalised by an entropy term. For example, the indifference price at time zero of a claim with $\mathcal{F}_T$-measurable payoff F is

$$p_0 = \sup_{\mathbb{Q} \in \mathbf{M}_f} \left[\mathbb{E}^{\mathbb{Q}}[F] - \frac{1}{\alpha} I_0(\mathbb{Q}|\mathbb{Q}^0) \right], \tag{1.1}$$

where $\alpha > 0$ is the risk aversion coefficient, $\mathbb{Q}^0$ is the minimal entropy martingale measure (MEMM) and $I_0(\mathbb{Q}|\mathbb{Q}^0)$ is the relative entropy between any ELMM $\mathbb{Q} \in \mathbf{M}_f$ and $\mathbb{Q}^0$.

Itô process example

- In an Itô process setting, and with $\varepsilon^2 = \alpha$, use of the Girsanov theorem renders the optimisation over measures in (1.1) to a problem in which the control is a drift perturbation to a (multi-dimensional) Brownian motion:

$$p := \sup_{\varphi \in \mathcal{A}(\mathbf{M}_f)} \mathbb{E}\left[F(X^{(\varepsilon)}) - \frac{1}{2}\int_0^T \|\varphi_t\|^2 \,\mathrm{d}t \right],$$

  where the state variable $X^{(\varepsilon)}$ is a perturbed process following

$$\mathrm{d}X_t^{(\varepsilon)} = a_t \,\mathrm{d}t + b_t(\,\mathrm{d}W_t + \varepsilon\varphi_t \,\mathrm{d}t),$$

  with $\varepsilon = 0$ corresponding to the dynamics under the MEMM $\mathbb{Q}^0$, $F(X^{(\varepsilon)})$ is a functional of the paths of $X^{(\varepsilon)}$, and a, b are adapted processes.
- Note: we fix the measure and consider a family $\{X^{(\varepsilon)}\}_{\varepsilon\in\mathbb{R}}$ of perturbed processes, as opposed to considering a fixed process under a family $\{\mathbb{Q}(\varepsilon)\}_{\varepsilon\in\mathbb{R}}$ of measures. This is for transparency and tractability.

Perturbations on Wiener space I

- For small ε, view the drift $\varepsilon\varphi$ as a perturbation to the Brownian paths on Wiener space. For $\varepsilon = 0$ the optimal control is zero, and we suppose that the optimal control for small ε will be a perturbation around zero.
- Then Malliavin calculus ideas arise in deriving an asymptotic expansion for the value function, valid for small ε. The power of this approach is that we can obtain results in non-Markovian models and for quite general path-dependent payoffs.
- We will differentiate the objective function of the control problem with respect to ε at $\varepsilon = 0$, and ultimately obtain an approximation of the value function for small ε.
- This uses Bismut's (1981) approach to the Malliavin calculus, which exploits the Girsanov theorem to translate a drift adjustment into to a measure change, in order to perform differentiation on path space.

Perturbations on Wiener space II

- Related work by Boué and Dupuis (1998) treats entropy-weighted control problems using similar variational principles on paths space, obtaining formulae of the form

$$-\log \mathbb{E}[\mathrm{e}^{-f(W)}] = \inf_{v} \mathbb{E}\left[\frac{1}{2}\int_0^T \|v_s\|^2\,\mathrm{d}s + f\left(W + \int_0^{\cdot} v_s\,\mathrm{d}s\right)\right].$$

  Bierkens and Kappen (2012) develop this further and obtain formulae for the optimal control as a Malliavin derivative of $f(W)$. Future work could seek to relate our results to these.
- We will not require our functional F to be Malliavin-differentiable, and will comment on what happens if it is.
- The idea of using Bismut's approach for asymptotics of stochastic control problems in finance is due to Davis (2006), for indifference pricing in a two-dimensional lognormal basis risk model, and for a path-independent claim.

Perturbations on Wiener space III

- Davis' method was neglected subsequently, as PDE methods (Zariphopoulou (2001), Henderson (2002), Monoyios (2004)) gave a closed-form nonlinear expectation representation for the indifference price, to which asymptotic analysis was easily applicable.
- Here, we resurrect Davis' idea, and show it can work in multi-dimensional Itô markets, with no Markov assumption, and for payoffs which are quite general path-dependent functionals.
- For exponential indifference valuation, BSDE and/or BMO techniques can also yield risk-aversion asymptotics (Becherer (2003), Mania and Schweizer (2005), Kallsen and Rheinländer (2011)). The salient point in this talk is the methodology, applied to a generic entropy-weighted control problem.

Directional derivative on Wiener space

- Take a canonical basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$.
- $\Omega = C_0([0,T]; \mathbb{R}^m)$, continuous functions $\omega : [0,T] \to \mathbb{R}^m$, with $\omega(0) = 0$.
- $\mathbb{P}$ is Wiener measure.
- $\{W_t(\omega) := \omega(t)\}_{t \in [0,T]}$ is m-dimensional Brownian motion.
- Given a functional $F(W)$ of the Brownian paths, an $\mathcal{F}_T$-measurable map $F : \Omega \to \mathbb{R}$ satisfying $\mathbb{E}[F^2(W)] < \infty$, we would like to define a directional derivative in the direction $\Phi \in \Omega$, with $\Phi := \int_0^{\cdot} \varphi_s \,\mathrm{d}s$:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} [F(W + \varepsilon\Phi)]|_{\varepsilon=0} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[F(W + \varepsilon\Phi) - F(W)],$$

  where one needs to make precise sense of the limit (we will do so in L_2).
- We will need the following condition on F: there exists a constant K such that for $\Phi \in \Omega$, and with $\|\cdot\|_\infty$ denoting the supremum norm $\|\omega(t)\|_\infty := \sup_{t \in [0,T]} \|\omega(t)\|$,

$$|F(W + \Phi) - F(W)| \le K \|\Phi\|_\infty. \tag{2.1}$$

Lemma (Directional derivative on Wiener space)

Let $F \equiv F(W)$ *be a square-integrable functional of the Brownian paths* W *on the Banach space* $\Omega = C_0([0,T];\mathbb{R}^m)$.
Let φ *be a bounded previsible process, with* $\Phi \in \Omega$ *defined by* $\Phi := \int_0^{\cdot} \varphi_s \, \mathrm{d}s$.
Then the map $\varepsilon \to \mathbb{E}[F(W+\varepsilon\Phi)]$ *is differentiable, with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}[F(W+\varepsilon\Phi)]\big|_{\varepsilon=0} = \mathbb{E}\left[F(W)(\varphi\cdot W)_T\right].$$

- Here, $(\varphi \cdot W)$ denotes the stochastic integral

$$\sum_{i=1}^{m}\int_0^T \varphi_t^i \, \mathrm{d}W_t^i \equiv \int_0^T \varphi_t \cdot \mathrm{d}W_t \equiv (\varphi\cdot W)_T.$$

- We do not need φ to be bounded, this is only for simplicity. We rely only on $(\varphi \cdot W)$ and $\mathcal{E}(-\varepsilon\varphi\cdot W)$ being martingales.

Relation to Malliavin derivative

- On the Hilbert space $H = L^2([0,T];\mathbb{R}^m)$, for $\varphi \in H$, the Cameron-Martin subspace $\mathcal{CM} \subset \Omega = C_0([0,T];\mathbb{R}^m)$ consists of functions $\Phi : [0,T] \to \mathbb{R}^m$ with square-integrable derivative φ:

$$\Phi_t := \int_0^t \varphi_s \,\mathrm{d}s, \quad \int_0^t \|\varphi_s\|^2 \,\mathrm{d}s < \infty, \quad 0 \le t \le T.$$

- If F is Malliavin-differentiable and $\Phi \in \mathcal{CM}$, the integration-by-parts formula is

$$\mathbb{E}\left[\int_0^T D_t F \cdot \varphi_t \,\mathrm{d}t\right] = \mathbb{E}\left[F \int_0^T \varphi_t \cdot \mathrm{d}W_t\right].$$

- So in this case the directional derivative also takes on the form above. But the directional derivative lemma is valid when F is not necessarily Malliavin-differentiable and for previsible φ such that $\mathbb{E}\left[\int_0^T \|\varphi_t\|^2 \,\mathrm{d}t\right] < \infty$.

Proof of Directional Derivative Lemma I

For φ bounded, previsible, and $\varepsilon \in \mathbb{R}$, define the exponential martingale

$$M_t^{(\varepsilon)} := \mathcal{E}(-\varepsilon\varphi \cdot W)_t$$
$$:= \exp\left(-\varepsilon \int_0^t \varphi \cdot \mathrm{d}W_s - \frac{1}{2}\varepsilon^2 \int_0^t \|\varphi_s\|^2 \,\mathrm{d}s\right), \quad 0 \le t \le T,$$

and the probaility measure $\mathbb{P}^{(\varepsilon)}$ by $\mathrm{d}\mathbb{P}^{(\varepsilon)} = M_T^{(\varepsilon)}\,\mathrm{d}\mathbb{P}$.
By Girsanov, $W^{(\varepsilon)} := W + \varepsilon\Phi$ is Brownian motion under $\mathbb{P}^{(\varepsilon)}$, so that

$$\mathbb{E}[F(W)] = \mathbb{E}^{(\varepsilon)}[F(W+\varepsilon\Phi)] = \mathbb{E}[M_T^{(\varepsilon)} F(W+\varepsilon\Phi)]. \tag{2.2}$$

This invarance principle underlies Bismut's approach to the Malliavin calculus.
Re-write (2.2) as

$$\mathbb{E}\left[\frac{1}{\varepsilon}(F(W+\varepsilon\Phi) - F(W))\right] + \mathbb{E}\left[\frac{1}{\varepsilon}(M_T^{(\varepsilon)} - 1)F(W)\right]$$
$$+ \quad \mathbb{E}\left[\frac{1}{\varepsilon}(F(W+\varepsilon\Phi) - F(W))(M_T^{(\varepsilon)} - 1)\right] = 0. \tag{2.3}$$

Proof of Directional Derivative Lemma II

Differentiate $\mathbb{E}[F(W+\varepsilon\Phi)]$ with respect to ε at $\varepsilon=0$ by considering what happens when we let $\varepsilon\to 0$ in (2.3). The last term is bounded by $K\|\Phi\|_\infty\mathbb{E}[|M_T^{(\varepsilon)}-1|]$, so tends to zero. Neglecting this term, we compute, using the square-integrability of F and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left(\mathbb{E}\left[\frac{1}{\varepsilon}(F(W+\varepsilon\Phi)-F(W))\right]-\mathbb{E}[F(W)(\varphi\cdot W)_T]\right)^2\\
= &\left(\mathbb{E}\left[\left(\frac{1}{\varepsilon}(M_T^{(\varepsilon)}-1)+(\varphi\cdot W)_T\right)F(W)\right]\right)^2\\
\le &\; C\mathbb{E}\left[\left(\frac{1}{\varepsilon}(M_T^{(\varepsilon)}-1)+(\varphi\cdot W)_T\right)^2\right],
\end{aligned}$$

which converges to zero as $\varepsilon\to 0$, using the well-known result that

$$\frac{1}{\varepsilon}(M_T^{(\varepsilon)}-1)\to -(\varphi\cdot W)_T,\quad \text{in } L_2,\text{ as } \varepsilon\to 0. \tag{2.4}$$

□

Remarks I

- If we place more structure on F we can illustrate the relation with the Malliavin derivative. Make the following assumption:
- Suppose there exists a kernel $\partial F(\omega;\cdot) \equiv \partial F(W;\cdot) : \Omega \to \mathbb{M}$, where $\mathbb{M}$ is the set of m-dimensional finite Borel measures on $[0, T]$, such that for $\Phi \in \Omega$, we have a directional derivative operator $\mathcal{D}_\Phi$ satisfying

$$\mathcal{D}_\Phi F(W) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F(W + \varepsilon\Phi) - F(W) \right) = \int_0^T \Phi_t \cdot \partial F(W; \mathrm{d}t). \qquad (2.5)$$

This condition is automatically satisfied if F is Fréchet differentiable, for in that case we have

$$F(W + \varepsilon\Phi) - F(W) = \varepsilon \int_0^T \Phi_t \cdot F'(W; \mathrm{d}t) + o(|\varepsilon| \|\Phi\|_\infty),$$

where the Fréchet derivative $F'(W;\cdot)$ is a bounded linear functional on Ω, that is, a measure (and hence an element of the dual space Ω^*). So in this case $\partial F \equiv F'$. But there are functionals where differentiablity fails but (2.5) holds (see Rogers and Williams).

Remarks II

- If $\Phi = \int_0^{\cdot} \varphi_s \, \mathrm{d}s$, then letting $\varepsilon \to 0$ in (2.3) as before, we get

$$\mathbb{E}\left[\int_0^T \Phi_t \cdot \partial F(W; \mathrm{d}t)\right] = \mathbb{E}\left[F(W)\int_0^T \varphi_t \cdot \mathrm{d}W_t\right],$$

that is

$$\mathbb{E}\left[\int_0^T \partial F(W;(t,T]) \cdot \varphi_t \, \mathrm{d}t\right] = \mathbb{E}\left[F(W)\int_0^T \varphi_t \cdot \mathrm{d}W_t\right]. \tag{2.6}$$

If F is Malliavin-differentiable, and $\Phi \in \mathcal{CM} \subset \Omega$, then the directional derivative $\mathcal{D}_\Phi F$ exists in $L_2(\mathbb{P})$ and is related to DF via

$$\mathcal{D}_\Phi F(W) = \int_0^T D_t F(W) \cdot \varphi_t \, \mathrm{d}t,$$

so in this case we have

$$\partial F(W;(t,T]) = D_t F(W), \quad t \in [0,T].$$

and (2.6) is the integration-by-parts formula.

Remainder term

Denoting $\|\varphi\|_2^2 := \int_0^T \|\varphi_t\|^2 \,\mathrm{d}t$, the Directional Derivative Lemma implies that

$$\mathbb{E}[F(W+\varepsilon\Phi) - F(W) - \varepsilon F(W)(\varphi \cdot W)_T] \sim O(\varepsilon^2 \|\varphi\|_2^2).$$

So in particular, if $\varphi = c\widetilde{\varphi}$ for some fixed $\widetilde{\varphi}$ and $c \in \mathbb{R}$, then

$$\mathbb{E}[F(W+\varepsilon\Phi) - F(W) - \varepsilon F(W)(\varphi \cdot W)_T] \sim O(c^2 \varepsilon^2). \tag{2.7}$$

Application to stochastic control

- On a canonical basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, a state variable $X^{(\varepsilon)} \in \mathbb{R}^m$ follows
$$\mathrm{d}X_t^{(\varepsilon)} = a_t\,\mathrm{d}t + b_t(\,\mathrm{d}W_t + \varepsilon\varphi_t\,\mathrm{d}t). \tag{3.1}$$
Here, a, b are adapted processes, and φ is such that $(\varphi \cdot W)$ is a martingale.
- A square-integrable random variable $F(X^{(\varepsilon)})$ is a functional of the paths of $X^{(\varepsilon)}$.
- The control problem is
$$p := \sup_{\varphi\in\mathcal{A}(\mathbf{M}_f)} \mathbb{E}\left[F(X^{(\varepsilon)}) - \frac{1}{2}\int_0^T \|\varphi_t\|^2\,\mathrm{d}t\right], \tag{3.2}$$
We suppose that, for small ε, the optimal control φ^* will be small.
- We expand the objective functional in (3.2) about $\varepsilon = 0$, considering $F(X^{(\varepsilon)})$ as a functional of the perturbed Brownian motion $W + \varepsilon\int_0^{\cdot}\varphi_s\,\mathrm{d}s$ and applying the Directional Derivative Lemma.

Theorem

Let $\varepsilon \in \mathbb{R}$ *be a small parameter. On the canonical basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, *define an* m*-dimensional Brownian motion* W. *Let* $\Phi := \int_0^{\cdot} \varphi_s \, \mathrm{d}s \in \Omega$ *be such that* $\mathbb{E}[\int_0^T \|\varphi_t\|^2 \, \mathrm{d}t] < \infty$. *Denote the set of such* φ *by* $\mathcal{A}(\mathbf{M}_f)$. *Let* $F(X^{(\varepsilon)})$ *be a square-integrable functional of the paths of the preturbed state process* $X^{(\varepsilon)}$, *which follows (3.1). The control problem with value function (3.2) has asymptotic value given by*

$$p = \mathbb{E}[F(X^{(0)})] + \frac{1}{2}\varepsilon^2 \mathbb{E}\left[\int_0^T \|\psi_t\|^2 \, \mathrm{d}t\right] + O(\varepsilon^4),$$

where ψ *is the integrand in the martingale representation of* $F(X^{(0)})$ *for* $\varepsilon = 0$:

$$F(X^{(0)}) = \mathbb{E}[F(X^{(0)})] + \int_0^T \psi_t \cdot \mathrm{d}W_t. \tag{3.3}$$

Proof (sketch) I

- Differentiate $\mathbb{E}[F(X^{(\varepsilon)})]$ with respect to ε at $\varepsilon = 0$, and invoke the martingale representation (3.3) of $F(X^{(0)})$. This gives

$$\begin{aligned} & \mathbb{E}\left[F(X^{(\varepsilon)}) - \frac{1}{2}\int_0^T \|\varphi_t\|^2 \, \mathrm{d}t\right] \\ = \quad & \mathbb{E}\left[F(X^{(0)}) + \int_0^T \left(\varepsilon\psi_t \cdot \varphi_t - \frac{1}{2}\|\varphi_t\|^2\right) \, \mathrm{d}t\right] + o(\varepsilon). \end{aligned}$$

Maximise over φ by choosing $\varphi = \widehat{\varphi}$, given by

$$\widehat{\varphi} := \varepsilon\psi, \tag{3.4}$$

to give

$$\mathbb{E}[F(X^{(0)})] + \frac{1}{2}\varepsilon^2\mathbb{E}\left[\int_0^T \|\psi_t\|^2 \, \mathrm{d}t\right] + O(\varepsilon^4).$$

The remainder term is of order ε^4 due to (2.7).

Proof (sketch) II

- We have maximised an approximation of the objective function. We need to check that the result does indeed constitute an approximation to the full control problem, to the same order in ε.
- In simple terms, we have written a function $J(\varepsilon, \varphi)$ as

$$J(\varepsilon, \varphi) = g(\varepsilon, \varphi) + O(\varepsilon^2 \varphi^2),$$

where

$$g(\varepsilon, \varphi) = J(0,0) + \varepsilon\varphi\psi - \frac{1}{2}\varphi^2.$$

Maximising g with respect to φ gives $\varphi = \widehat{\varphi} = \varepsilon\psi$, and then

$$J(\varepsilon, \widehat{\varphi}) = g(\varepsilon, \widehat{\varphi}) + O(\varepsilon^4) = J(0,0) + \frac{1}{2}\varepsilon^2\psi^2 + O(\varepsilon^4).$$

But we need to show that

$$\sup_{\varphi} J(\varepsilon, \varphi) = g(\varepsilon, \widehat{\varphi}) + O(\varepsilon^4).$$

Proof (sketch) III

- Consider maximising over φ, a smooth function $J(\varepsilon, \varphi)$ given by

$$J(\varepsilon, \varphi) := f(x + \varepsilon\varphi) - \frac{1}{2}\varphi^2.$$

The optimiser satisfies

$$\varphi^* = \varepsilon f'(x + \varepsilon\varphi^*), \tag{3.5}$$

and for $\varepsilon = 0$, $\varphi^* = 0$. If we write

$$\varphi^* = \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \varepsilon^3\varphi^{(3)} + \varepsilon^4\varphi^{(4)} + O(\varepsilon^5\varphi^{(5)}),$$

then using this in (3.5) along with a Taylor expansion gives

$$\varphi^* = \varepsilon f'(x)(1 + \varepsilon^2 f''(x)) + O(\varepsilon^5).$$

Then the maximum has approximate value given by

$$J(\varepsilon, \varphi^*) = f(x) + \frac{1}{2}\varepsilon^2(f'(x))^2 + O(\varepsilon^4).$$

Proof (sketch) IV

But this is the same value as is obtained by maximising the linear in ε approximation to $J(\varepsilon, \varphi)$.

$$J(\varepsilon, \varphi) = f(x) + \varepsilon\varphi f'(x) - \frac{1}{2}\varphi^2 + O(\varepsilon^2\varphi^2),$$

which is maximised by $\widehat{\varphi} = \varepsilon f'(x)$, yielding

$$J(\varepsilon, \widehat{\varphi}) = f(x) + \frac{1}{2}\varepsilon^2(f'(x))^2 + O(\varepsilon^4),$$

so that $J(\varepsilon, \varphi^*) = J(\varepsilon, \widehat{\varphi})$ to order ε^2, with the error being of order ε^4 in both cases.

Proof (sketch) V

- In the stochastic control problem, perform a similar (but more delicate) analysis. The objective functional can be written as

$$\mathbb{E}\left[F(W) + \int_0^T \left(\varepsilon \partial F\left(W + \varepsilon \int_0^{\cdot} \varphi_s \,\mathrm{d}s; (t, T]\right) \cdot \varphi_t - \frac{1}{2}\|\varphi_t\|^2\right) \mathrm{d}t\right],$$

so the optimal control for the full problem satisfies

$$\varphi_t^* = \varepsilon \partial F\left(W + \varepsilon \int_0^{\cdot} \varphi_s^* \,\mathrm{d}s; (t, T]\right), \quad 0 \leq t \leq T, \qquad (3.6)$$

which is the analogue of (3.5).

- If F is Malliavin differentiable and we restrict to controls such that $\int_0^{\cdot} \varphi_s \,\mathrm{d}s \in \mathcal{CM}$, then (3.6) becomes

$$\varphi_t^* = \varepsilon D_t F\left(W + \varepsilon \int_0^{\cdot} \varphi_s^* \,\mathrm{d}s\right), \quad 0 \leq t \leq T.$$

Proof (sketch) VI

- Develop a Taylor expansion of the RHS of (3.6) by using variational principles, considering perturbations of φ^*.
- Use this to show that using the approximate control $\widehat{\varphi}$ in (3.4) does indeed give the approximation to the full problem, to precision ε^2, with error term of order ε^4.

$\square$

Dual control representation of indifference price I

- On $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}\mathbb{P})$, the discounted prices of d stocks are modelled by a positive locally bounded semi-martingale S.
- An agent trades S and maximises utility of terminal wealth, with the liability of an $\mathcal{F}_T$-measurable claim payoff F:

$$u_t^F(x_t) := \operatorname*{ess\,sup}_{\theta\in\Theta_t} \mathbb{E}\left[-\mathrm{e}^{-\alpha\left(x_t + \int_t^T \theta_u \cdot \mathrm{d}S_u - F\right)} \middle| \mathcal{F}_t \right], \quad 0 \le t \le T, \tag{4.1}$$

Denote the optimiser by θ^F. Set $F \equiv 0$ to recover corresponding objects in the problem without the claim.

- The utility indifference price process for the claim is $p(\alpha)$ defined by

$$u_t^F(x_t + p_t(\alpha)) = u_t^0(x_t), \quad 0 \le t \le T. \tag{4.2}$$

To invoke duality, introduce the conditional relative entropy process between $\mathbb{Q} \in \mathbf{M}_f$ and $\mathbb{P}$, as

$$I_t(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}[\log Z_{t,T}^{\mathbb{Q}}|\mathcal{F}_t], \quad 0 \le t \le T. \tag{4.3}$$

Dual control representation of indifference price II

The dual problem to (4.1) is defined by

$$I_t^F := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathbf{M}_f} \left[I_t(\mathbb{Q}|\mathbb{P}) - \alpha\mathbb{E}^{\mathbb{Q}}[F|\mathcal{F}_t]\right], \quad 0 \le t \le T. \tag{4.4}$$

Denote the optimiser in (4.4) by $\mathbb{Q}^F$.

Lemma

The indifference price process is given by the dual stochastic control representation

$$p_t(\alpha) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathbf{M}_f} \left[\mathbb{E}^{\mathbb{Q}}[F|\mathcal{F}_t] - \frac{1}{\alpha} I_t(\mathbb{Q}|\mathbb{Q}^0)\right], \quad 0 \le t \le T.$$

This follows from (a dynamic version of) the classical dual representation of indifference prices (Delbaen *et al* (2002), Becherer (2003)):

$$p_t(\alpha) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathbf{M}_f} \left[\mathbb{E}^{\mathbb{Q}}[F|\mathcal{F}_t] - \frac{1}{\alpha}\left(I_t(\mathbb{Q}|\mathbb{P}) - I_t(\mathbb{Q}^0|\mathbb{P})\right)\right], \quad 0 \le t \le T, \tag{4.5}$$

Dual control representation of indifference price III

combined with the following result:

Proposition (Entropic distances are co-linear)

For $\mathbb{Q} \in \mathbf{M}_f$, the conditional entropy process I satisfies

$$I_t(\mathbb{Q}|\mathbb{P}) - I_t(\mathbb{Q}^0|P) = I_t(\mathbb{Q}|\mathbb{Q}^0), \quad 0 \leq t \leq T. \tag{4.6}$$

These results all stem from a dynamic version of the fundamental results of Grandits and Rheinländer (2002), Kabanov and Stricker (2002), linking the optimal strategy θ^F to the minimiser $\mathbb{Q}^F$ in the dual problem.

Multi-dimensional Itô market

- On $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T} \mathbb{P})$, with an m-dimensional Brownian motion W, $d < m$ stock prices $S = (S^1, \dots, S^d)^\top$ follow

$$\mathrm{d}S_t = \mathrm{diag}_d(S_t)[\mu_t^S \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t], \tag{5.1}$$

The d-dimensional vector μ^S and the $(d \times m)$ matrix σ are $\mathbb{F}$-progressively measurable processes, such that the m-dimensional relative risk process

$$\lambda_t := \sigma_t^\top (\sigma_t \sigma_t^\top)^{-1} \mu_t^S, \quad 0 \le t \le T, \tag{5.2}$$

is well-defined.

- A vector $Y = (Y^1, \dots, Y^{m-d})^\top$ of $(m - d)$ non-traded factors follows

$$\mathrm{d}Y_t = \mathrm{diag}_{m-d}(Y_t)[\mu_t^Y \, \mathrm{d}t + \beta_t \, \mathrm{d}W_t].$$

Local martingale measures

- Measures $\mathbb{Q} \in \mathbf{M}_f$ have density processes with respect to $\mathbb{P}$ of the form

$$Z_t^{\mathbb{Q}} = \mathcal{E}(-q \cdot W)_t, \quad 0 \le t \le T, \tag{5.3}$$

for some m-dimensional process q such that $Z^{\mathbb{Q}}$ is a $\mathbb{P}$-martingale, q satisfies

$$\mu_t^S - \sigma_t q_t = \mathbf{0}_d, \quad 0 \le t \le T, \tag{5.4}$$

and the finite entropy condition gives

$$\Lambda^{\mathbb{Q}} := (q \cdot W^{\mathbb{Q}}) \quad \text{is a } \mathbb{Q}\text{-martingale, for all } \mathbb{Q} \in \mathbf{M}_f. \tag{5.5}$$

The market is incomplete, so there will be an infinite number of solutions q to the equations (5.4). For $q = \lambda$ we obtain the minimal martingale measure $\mathbb{Q}_M$, while the density process of the MEMM $\mathbb{Q}^0$ is $Z^{\mathbb{Q}^0} = \mathcal{E}(-q^0 \cdot W)$, for some integrand q^0.

$\mathbb{Q}$ as a perturbation around $\mathbb{Q}^0$ I

- We can write the $\mathbb{Q}$-dynamics of Y as

$$\mathrm{d}Y_t = \mathrm{diag}_{m-d}(Y_t)[(\mu_t^Y - \beta_t q_t^0)\,\mathrm{d}t + \beta_t(\,\mathrm{d}W_t^{\mathbb{Q}} - (q_t - q_t^0)\,\mathrm{d}t)].$$

  The point of this representation is that the $\mathbb{Q}$-dynamics of Y can be considered as a perturbation of the $\mathbb{Q}^0$-dynamics.

- The entropy process between $\mathbb{Q}$ and $\mathbb{Q}^0$ is

$$I_t(\mathbb{Q}|\mathbb{Q}^0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_t^T \|q_u - q_u^0\|^2\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T. \tag{5.6}$$

$\mathbb{Q}$ as a perturbation around $\mathbb{Q}^0$ II

- Introduce, for some small parameter ε, a parametrised family of measures $\{\mathbb{Q}(\varepsilon)\}_{\varepsilon\in\mathbb{R}}$, such that

$$\mathbb{Q} \equiv \mathbb{Q}(\varepsilon), \quad \mathbb{Q}^0 \equiv \mathbb{Q}(0), \tag{5.7}$$

  and set

$$q - q^0 =: -\varepsilon\varphi, \tag{5.8}$$

  for some process φ. Since both q and q^0 satisfy (5.4), we have

$$\sigma\varphi = \mathbf{0}_d. \tag{5.9}$$

  Denote by $\mathcal{A}(\mathbf{M}_f)$ the set of such φ which correspond to $\mathbb{Q} \in \mathbf{M}_f$, and also define the process $\Phi := \int_0^{\cdot} \varphi_s \, \mathrm{d}s$.

- The $\mathbb{Q}(\varepsilon)$-dynamics of the state variables S, Y in this notation are then

$$\begin{aligned} \mathrm{d}S_t &= \mathrm{diag}_d(S_t)\sigma_t \, \mathrm{d}W_t^{\mathbb{Q}(\varepsilon)}, \\ \mathrm{d}Y_t &= \mathrm{diag}_{m-d}(Y_t)[(\mu_t^Y - \beta_t q_t^0)\, \mathrm{d}t + \beta_t(\mathrm{d}W_t^{\mathbb{Q}(\varepsilon)} + \varepsilon\varphi_t)\, \mathrm{d}t]. \end{aligned}$$

  With state variable $X := (S, Y)^\top$, we have dynamics of the form (3.1).

Orthogonality between strategies and dual controls

The $\mathbb{Q}(\varepsilon)$-dynamics of S, along with the constraint (5.9), lead to the following orthogonality result.

Lemma

Consider integrands $\theta^{(\varepsilon)}, \varphi$ such that $(\theta^{(\varepsilon)} \cdot S)$ is a $\mathbb{Q}(\varepsilon)$-martingale and φ satisfies (5.9). Then the stochastic integrals $(\theta^{(\varepsilon)} \cdot S)$ and $(\varphi \cdot W^{\mathbb{Q}(\varepsilon)})$ are orthogonal $\mathbb{Q}(\varepsilon)$-martingales. That is,

$$\mathbb{E}^{\mathbb{Q}(\varepsilon)}[(\theta^{(\varepsilon)} \cdot S)_T(\varphi \cdot W^{\mathbb{Q}(\varepsilon)})_T] = 0.$$

Note this holds for $\varepsilon \in \mathbb{R}$, and in particular for $\varepsilon = 0$.

Stochastic control problem for indifference price I

- Let F be an $\mathcal{F}_T$-measurable square-integrable functional of the paths of $X = (S, Y)$, and hence of the Brownian paths, representing the payoff of a European claim.
- The Galtchouk-Kunita-Watanabe decomposition of F under $\mathbb{Q}(\varepsilon)$ is

$$F = \mathbb{E}^{\mathbb{Q}(\varepsilon)}[F] + (\theta^{(\varepsilon)} \cdot S)_T + (\xi^{(\varepsilon)} \cdot W^{\mathbb{Q}(\varepsilon)})_T, \tag{5.10}$$

for some integrands $\theta^{(\varepsilon)}, \xi^{(\varepsilon)}$, such that the stochastic integrals in (5.10) are orthogonal $\mathbb{Q}(\varepsilon)$-martingales, so we have

$$\mathbb{E}^{\mathbb{Q}(\varepsilon)}[(\theta^{(\varepsilon)} \cdot S)_T(\xi^{(\varepsilon)} \cdot W^{\mathbb{Q}(\varepsilon)})_T] = 0.$$

On using (5.6) and (5.8), the indifference price process, from its dual stochastic control representation, is given as

$$p_t(\alpha) = \sup_{\varphi \in \mathcal{A}(\mathbf{M}_f)} \mathbb{E}^{\mathbb{Q}(\varepsilon)} \left[F - \frac{\varepsilon^2}{2\alpha} \int_t^T \|\varphi_u\|^2 \, \mathrm{d}u \middle| \mathcal{F}_t \right], \quad 0 \le t \le T.$$

Stochastic control problem for indifference price II

If we choose

$$\varepsilon^2 = \alpha, \tag{5.11}$$

then we get a control problem of the form

$$p_t(\alpha) = \sup_{\varphi \in \mathcal{A}(\mathbf{M}_f)} \mathbb{E}^{\mathbb{Q}(\varepsilon)} \left[F - \frac{1}{2} \int_t^T \|\varphi_u\|^2 \, \mathrm{d}u \middle| \mathcal{F}_t \right], \quad 0 \le t \le T. \tag{5.12}$$

subject to $\mathbb{Q}(\varepsilon)$-dynamics of S, Y

$$\begin{aligned} \mathrm{d}S_t &= \mathrm{diag}_d(S_t)\sigma_t \, \mathrm{d}W_t^{\mathbb{Q}(\varepsilon)}, \\ \mathrm{d}Y_t &= \mathrm{diag}_{m-d}(Y_t)[(\mu_t^Y - \beta_t q_t^0)\,\mathrm{d}t + \beta_t(\,\mathrm{d}W_t^{\mathbb{Q}(\varepsilon)} + \varepsilon\varphi_t\,\mathrm{d}t)], \end{aligned}$$

such that $\varepsilon = 0$ gives the minimal entropy measure.

Theorem

Let the payoff of the claim, F, be a square-integrable functional of the paths of S, Y. For small risk aversion α, the indifference price process of the claim has the asymptotic expansion

$$p_t(\alpha) = \mathbb{E}^{\mathbb{Q}^0}[F|\mathcal{F}_t] + \frac{1}{2}\alpha\mathbb{E}^{\mathbb{Q}^0}\left[\int_t^T \|\xi_u^{(0)}\|^2 \,\mathrm{d}u \middle| \mathcal{F}_t\right] + O(\alpha^2), \quad 0 \le t \le T,$$

where $\mathbb{Q}^0$ is the minimal entropy martingale measure, and $\xi^{(0)}$ is the process in the Kunita-Watanabe decomposition (5.10) of the claim, under $\mathbb{Q}(0) \equiv \mathbb{Q}^0$.

Remark

Using the Kunita-Watanabe decomoposition we can write the result as

$$\begin{aligned} p_t(\alpha) &= \mathbb{E}^{\mathbb{Q}^0}[F|\mathcal{F}_t] \\ &+ \frac{1}{2}\alpha\left(\mathrm{var}^{\mathbb{Q}^0}[F|\mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}^0}\left[\int_t^T \|\theta_u^{(0)}\|^2\,\mathrm{d}[S]_u\,\middle|\,\mathcal{F}_t\right]\right) + O(\alpha^2), \end{aligned}$$

for $t \in [0, T]$, which highlights the mean-variance structure of the asymptotic representation.

The Itô process framework here encompasses many well-known basis risk models, and some less well-known examples, such as those with:

- stochastic correlation and/or stochastic volatility,
- drift uncertainty(partial information),

and multi-factor stochastic volatility models.

Entropy minimisation in stochastic volatility model

Theorem

In the stochastic volatility model

$$\begin{aligned} \mathrm{d}S_t &= \sigma(Y_t)S_t\left(\lambda(Y_t)\,\mathrm{d}t + \mathrm{d}W_t\right), \\ \mathrm{d}Y_t &= a(Y_t)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}\widetilde{W}_t, \end{aligned}$$

the relative entropy between the minimal entropy martingale measure $\mathbb{Q}_E$ and $\mathbb{P}$, in the limit that $1-\rho^2 \approx 1$, is given as

$$I_0(\mathbb{Q}_E|\mathbb{P}) = I_0(\mathbb{Q}_M|\mathbb{P}) - \frac{1}{8}(1-\rho^2)\mathrm{var}^{\mathbb{Q}_M}[K_T] + O((1-\rho^2)^2),$$

where $\mathbb{Q}_M$ is the minimal martingale measure and K is the mean-variance trade-off process

$$K_t := \int_0^t \lambda^2(Y_u)\,\mathrm{d}u, \quad 0 \le t \le T.$$

Remark

In MM (2006,2007), Esscher transform relations between $\mathbb{Q}_E$ and $\mathbb{Q}_M$ are derived:

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}_E}{\mathrm{d}\widetilde{\mathbb{Q}}_M} = \frac{\exp(\theta K_T)}{\mathbb{E}^{\widetilde{Q}_M}[\exp(\theta K_T)]},$$

where $\theta = -\frac{1}{2}(1-\rho^2)$ and $\widetilde{\mathbb{Q}}_E, \widetilde{\mathbb{Q}}_M$ are the projections of $\mathbb{Q}_E, \mathbb{Q}_M$ onto $\widetilde{\mathcal{F}}_T = \sigma(\widetilde{W}_t; 0 \leq t \leq T)$, and it is an exercise in asymptotic analysis to see that those results are consistent with this theorem.

Extensions

- Lévy state dynamics.
- Other types of variation applied to paths (Cont, Fournié, Dupire).

The paper is undergoing a revision and will appear shortly in a new guise at www.maths.ox.ac.uk/~monoyios