Heat Kernel Framework for Asset Pricing in Finite Time

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Asset pricing

The main modelling ingredients of an asset pricing framework are:

(A) The stream of cash-flows.

(B) The pricing kernel, i.e. interest rates and market price of risk.

(C) The market filtration, i.e. the market information.

The aim is to construct coherent asset pricing models.

Such models shall be constructed in a logically-connected and consistent manner, shall have a natural design, and shall form a unified whole.

A modern asset pricing framework should be coherent across asset classes, and ensure consistency for the pricing of assets and the risk management of asset portfolios.

We model a financial market by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ denotes the market filtration, and $\mathbb{P}$ is the real probability measure.

The price S_t at time $t \geq 0$ of a dividend-paying asset with a terminal random cash flow S_T at the fixed time $T \geq t$ is given by:

$$S_t = \frac{1}{\pi_t} \mathbb{E}^{\mathbb{P}} \left[\pi_T S_T + \int_t^T \pi_u D_u \mathrm{d}u \,\middle|\, \mathcal{F}_t \right]. \tag{1}$$

The pricing kernel process $\{\pi_t\}_{0 \leq t}$ determines the inter-temporal relationship between the future asset cash flow at T and the asset price at time t.

We consider a pricing kernel $\{\pi_t\}$ that is modelled by a function of the form

$$\pi_t = \pi(t, x). \tag{2}$$

The function $\pi(t, x)$ shall be deduced such that the pricing kernel is a positive $(\{\mathcal{F}_t\}, \mathbb{P})$-supermartingale.

Ideally, the constructed asset pricing framework allows for direct model calibrations to market data and other relevant information.

Pricing kernel models in finite time

We fix a finite time $U > 0$.

We introduce a process $\{X_t\}_{0\le t\le U}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ that generates the market filtration, that is $\mathcal{F}_t := \sigma\left(\{X_s\}_{0\le s\le t}\right)$.

Let $\{X_t\}$ have the Markov property with respect to $\{\mathcal{F}_t\}$.

We consider the following pricing kernel model:

$$\pi_t = \pi(t, X_t) = f_0(t) + f_1(t)\int_0^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) \,|\, X_t\right] w(t,u)\mathrm{d}u, \qquad (3)$$

where $f_0(t)$ and $f_1(t)$ are deterministic positive non-increasing functions, $F(t,x)$ is a positive measurable function, and $w(t,u)$ satisfies

$$w(t, u-s) \le w(t-s, u) \qquad (4)$$

for $s \le t \wedge u$, $t + u < U$.

It can be proven that the considered pricing kernel processes are indeed $(\{\mathcal{F}_t\}, \mathbb{P})$-supermartingales.

Bond price models and associated interest rates

By an application of the general pricing formula, we can calculate the price P_{tT} of a discount bond with maturity T, where $0 \leq t \leq T$:

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}\left[\pi_T \,|\, X_t\right]. \tag{5}$$

We insert the pricing kernel model

$$\pi_t = f_0(t) + f_1(t) \int_0^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) \,|\, X_t\right] w(t,u)\mathrm{d}u, \tag{6}$$

and obtain

$$P_{tT} = \frac{f_0(T) + f_1(T) \int_{T-t}^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) \,|\, X_t\right] w(T, u-T+t)\mathrm{d}u}{f_0(t) + f_1(t) \int_0^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) \,|\, X_t\right]\, w(t,u)\mathrm{d}u}. \tag{7}$$

For $0 \leq t < U$, the initial term structure is given by

$$P_{0t} = \frac{f_0(t) + f_1(t) \int_t^U \mathbb{E}\left[F(u, X_u) \,|\, X_0\right] w(t, u-t)\mathrm{d}u}{f_0(0) + f_1(0) \int_0^U \mathbb{E}\left[F(u, X_u) \,|\, X_0\right] \, w(0, u)\mathrm{d}u}. \tag{8}$$

We then solve for $f_0(t)$ to express it in terms of P_{0t}, set $f_0(0) = 1$ with no loss of generality, and plug the result into equation (7).

We obtain the following compact form:

$$P_{tT} = \frac{P_{0T} + y(T)\left(Y_{tT} - Y_{0T}\right)}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}, \tag{9}$$

where we define

$$Y_{tT} = Y(t, T, X_t) := \int_{T-t}^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) \,|\, X_t\right] w(T, u - T + t)\mathrm{d}u, \tag{10}$$

and

$$y(t) := \frac{f_1(t)}{1 + f_1(0) Y_{00}}. \tag{11}$$

The pricing kernel now takes the form

$$\pi_t = \pi_0 \left[P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)\right], \tag{12}$$

where

$$\pi_0 = 1 + f_1(0) Y_{00}. \tag{13}$$

We emphasize that the bond price and the pricing kernel are automatically calibrated to the initial term structure P_{0t}.

Assuming that the function of the bond price is differentiable in T, we can calculate the interest rate as follows:

$$\begin{aligned} r_t &= -\partial_T \ln\left(P_{tT}\right)\Big|_{T=t} \\ &= -\frac{\partial_t P_{0t} + \left(Y_{tt} - Y_{0t}\right)\left\{\partial_T\, y(T)\right\}_{T=t} + y(t)\left\{\left(\partial_T Y_{tT} - \partial_T Y_{0T}\right)\right\}_{T=t}}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}. \end{aligned} \tag{14}$$

The interest rate model is positive by construction, and is automatically calibrated to the initial term structure.

The additional degree of freedom $y(t)$ can be used to calibrate the bond price model (and thus the interest rate) to option data, e.g. caplets and swaptions.

Simulations of the interest rate dynamics $r_t = r(t, X_t)$ are straightforward, and they essentially involve the simulation of a Markov process.

Explicit asset pricing models

We can identify classes of pricing models with which bond prices, interest rates, risk premiums, and derivatives on interest rates can be calculated in (semi-) closed form.

Let $\{M_t\}_{0\le t<U}$ be an $\{\mathcal{F}_t\}$-adapted $\mathbb{P}$-martingale that induces a change-of-measure from $\mathbb{P}$ to an equivalent auxiliary probability measure $\mathbb{M}$.

Let $\{Y^{\mathbb{M}}_{tT}\}_{0\le t\le T<U}$ be defined by

$$Y^{\mathbb{M}}_{tT} = \int_{T-t}^{U-t} \mathbb{E}^{\mathbb{M}}\left[F(t+u, X_{t+u}) \,|\, X_t\right] w(T, u-T+t)\,\mathrm{d}u. \tag{15}$$

Then, the process

$$\pi_t = \pi_0 \left[P_{0t} + y(t)\left(Y^{\mathbb{M}}_{tt} - Y^{\mathbb{M}}_{0t}\right)\right] M_t \tag{16}$$

is a positive $(\{\mathcal{F}_t\}, \mathbb{P})$-supermartingale where

$$P_{0t} = \frac{f_0(t) + f_1(t)\, Y^{\mathbb{M}}_{0t}}{f_0(0) + f_1(0)\, Y^{\mathbb{M}}_{00}}, \qquad y(t) = \frac{f_1(t)}{1 + f_1(0) Y^{\mathbb{M}}_{00}}. \tag{17}$$

The discount bond price process is given by

$$P_{tT} = \frac{P_{0T} + y(T)\left(Y_{tT}^{\mathbb{M}} - Y_{0T}^{\mathbb{M}}\right)}{P_{0t} + y(t)\left(Y_{tt}^{\mathbb{M}} - Y_{0t}^{\mathbb{M}}\right)}. \tag{18}$$

Next, we consider $(\{\mathcal{F}_t\}, \mathbb{M})$-martingales $\{A_t\}$ and deterministic decreasing functions $b(t)$.

We find that for bond price and associated interest rate processes of the form

$$P_{tT} = \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t}, \qquad r_t = -\frac{\partial_t P_{0t} + A_t\,\partial_t b(t)}{P_{0t} + b(t)A_t}, \tag{19}$$

the prices of caplets and swaptions can be computed in (semi-)closed form.

Numerical root-finding may be necessary to calculate the price of in-the-money options that are driven by multivariate Markov processes.

Quadratic and exponential quadratic conditional Gaussian models

In order to provide explicit pricing models, we need to specify

(i) the Markov process $\{X_t\}_{0\leq t\leq U}$,

(ii) the positive integrable function $F(t,x)$, and

(iii) the weight function $w(t,u)$,

which together generate the dynamics of the pricing kernel.

Let us, for example, consider the following process:

$$L_{tU} = \sigma\, t\, X_U + \beta_{tU}, \tag{20}$$

where σ is a constant parameter, X_U is a random variable with *a priori* density $\mathbb{P}[X_U \in \mathrm{d}x] = p(x)\mathrm{d}x$, and $\{\beta_{tU}\}_{0\leq t\leq U}$ is an independent standard Brownian bridge.

It can be shown that $\{L_{tU}\}_{0\leq t\leq U}$ is a Markov process with respect to its natural filtration, and hence we can use such a process to generate the market filtration.

Quadratic class.

Let $F(t,x)=x^2$, and set $w(t,u)=U-t-u$.

Then, we have:

$$A_t=\frac{U}{(U-t)^2}\,L_{tU}^2-\frac{t}{U-t},\qquad b(t)=\frac{(U-t)^4\,f_1(t)}{4U\left[1+\frac{1}{12}f_1(0)U^3\right]}. \tag{21}$$

Exponential quadratic class.

Let $w(t,u)=(U-t-u)^{\eta-1/2}$ for $\eta>1/2$, and let

$$F(t,x)=\exp\left(\frac{x^2}{2(U-t-u)}\right). \tag{22}$$

Then, we have:

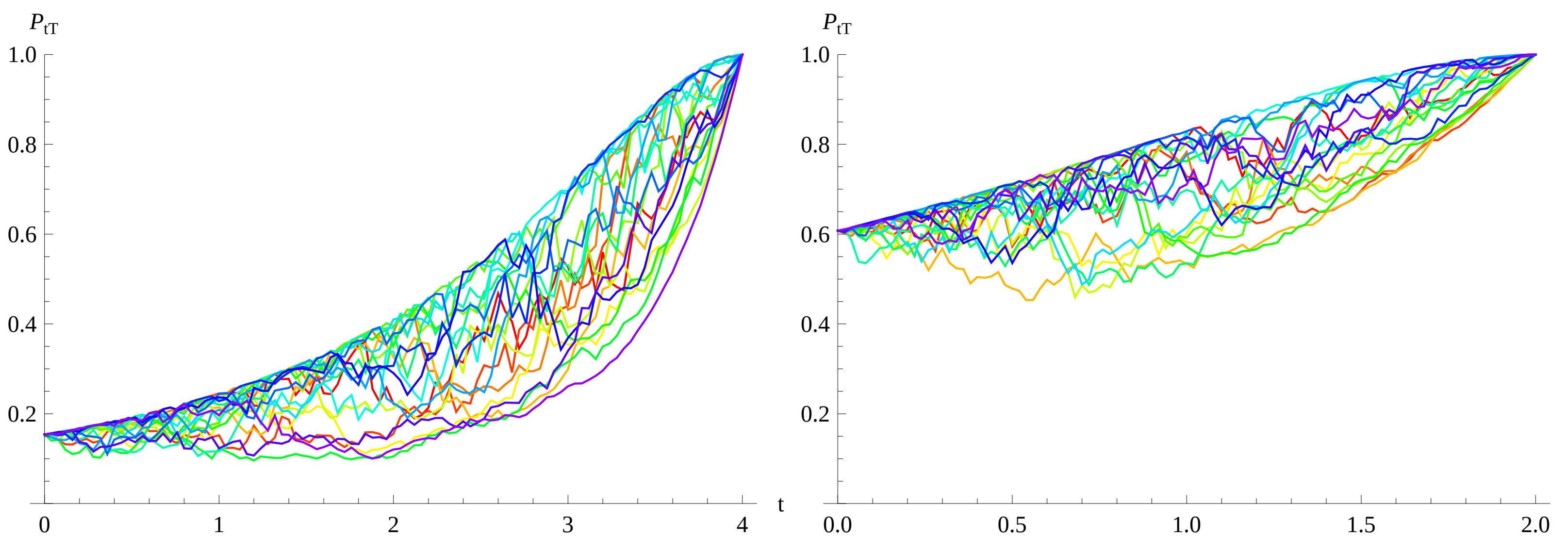
$$A_t=\sqrt{1-\frac{t}{U}}\,\exp\left(\frac{L_{tU}^2}{2(U-t)}\right)-1,\quad b(t)=\frac{(U-t)^{\eta-1/2}\,U^{1/2}\,f_1(t)}{1+f_1(0)U^{\eta}}. \tag{23}$$

The processes $\{A_t\}_{0\leq t<U}$ are $\mathbb{M}$-martingales where $\mathbb{M}$ is the measure under which $\{L_{tU}\}$ has the law of a Brownian bridge.

The bond price and the interest rate processes are thus given in explicit form by formulae (19). The quadratic models, for example, give

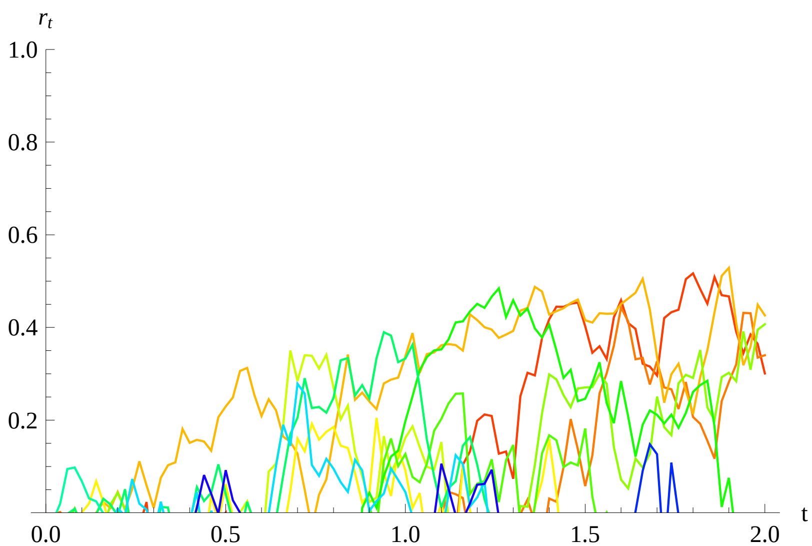
$$P_{tT} = \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t} = \frac{P_{0T} + \frac{(U-T)^4 f_1(T)}{4U\left[1+\frac{1}{12}f_1(0)U^3\right]}\left(\frac{U}{(U-t)^2}L_{tU}^2 - \frac{t}{U-t}\right)}{P_{0t} + \frac{(U-t)^4 f_1(t)}{4U\left[1+\frac{1}{12}f_1(0)U^3\right]}\left(\frac{U}{(U-t)^2}L_{tU}^2 - \frac{t}{U-t}\right)}. \tag{24}$$

Simulations of such bond price processes produce the following plots:

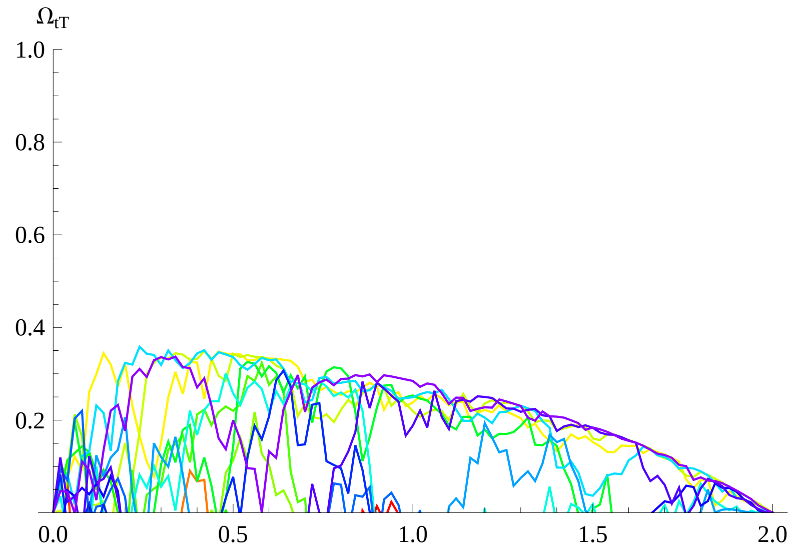


Plot 1: T=4, U=6. Plot 2: T=2, U=6.

For $T = 2$ and $U = 6$, the simulations of the associated quadratic models for the interest rate $\{r_t\}$ and the bond volatility $\{\Omega_{tT}\}$ show the following dynamics:



Interest rate simulation for T=2, U=6.



Bond volatility simulation for T=2, U=6.

Interest rate derivatives

The price of interest rate options can be calculated in (semi-)closed form. In the case that the form of the discount bond price process is

$$P_{tT} = \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t}, \tag{25}$$

the price at time 0 of a swaption contract with maturity t and swap rate K is

$$\begin{aligned} Sw_{0t} &= \frac{1}{\pi_0}\mathbb{E}\left[\pi_t\left(1 - P_{tT_n} - K\sum_{i=1}^{n} P_{tT_i}\right)^{+}\right], \\ &= \left(P_{0t} - P_{0T_n} - K\sum_{i=1}^{n} P_{0T_i}\right)\int_{a^S} p(a)\,\mathrm{d}a \\ &\qquad + \left[b(t) - b(T_n) - K\sum_{i=1}^{n} b\left(T_i\right)\right]\int_{a^S} a\,p(a)\,\mathrm{d}a, \end{aligned} \tag{26}$$

where $P[A_t \in \mathrm{d}a] = p(a)\mathrm{d}a$ and

$$a^S := \left\{a : \, a > \frac{K\sum_{i=1}^{n} P_{0T_i} - P_{0t} + P_{0T_n}}{b(t) - b\left(T_n\right) - K\sum_{i=1}^{n} b\left(T_i\right)}\right\}.$$

In the special case of an exponential quadratic pricing kernel model that is driven by a single Brownian random bridge, the price Sw_{0t} of the swaption is

$$\begin{aligned} Sw_{0t} \; &= \; \left(P_{0t} - P_{0T_n} - K \sum_{i=1}^{n} P_{0T_i}\right) N(-\nu) \\ &\quad + \left[b(t) - b\left(T_n\right) - K \sum_{i=1}^{n} b\left(T_i\right)\right] \left[N(\nu) - N\left(\nu \sqrt{1 - t/U}\right)\right], \end{aligned} \tag{27}$$

where $N(x)$ is the cumulative normal distribution function, and here

$$\nu := \sqrt{\frac{2U}{t} \ln \left[\left(1 - \frac{t}{U}\right)^{-1/2} \left(\frac{K \sum_{i=1}^{n} P_{0T_i} + P_{0T_n} - P_{0t}}{b(t) - b\left(T_n\right) - K \sum_{i=1}^{n} b\left(T_i\right)} + 1\right)\right]}. \tag{28}$$

Dynamical equation of the discount bond price process

We investigate the class of bond price processes of the form

$$P_{tT} = \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t} \tag{29}$$

in the case where

$$\mathrm{d}A_t = \nu_t \left(\mathrm{d}W_t + \vartheta_t \,\mathrm{d}t\right). \tag{30}$$

Here $\{\nu_t\}$ and $\{\vartheta_t\}$ are $\{\mathcal{F}_t\}$-adapted processes, and $\{W_t\}$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$-Brownian motion.

By an application of Ito's Lemma, we obtain the dynamical equation of the bond price process:

$$\frac{\mathrm{d}P_{tT}}{P_{tT}} = \left(r_t + \lambda_t \,\Omega_{tT}\right)\mathrm{d}t + \Omega_{tT}\,\mathrm{d}W_t. \tag{31}$$

where $\{r_t\}$ is the interest rate, $\{\lambda_t\}_{0\leq t<U}$ is the market price of risk, and $\{\Omega_{tT}\}_{0\leq t\leq T}$ is the bond volatility.

We have:

$$r_t = -\frac{\partial_t P_{0t} + A_t \partial_t b(t)}{P_{0t} + b(t)\, A_t}, \tag{32}$$

$$\lambda_t = \vartheta_t - \nu_t \frac{b(t)}{P_{0t} + b(t)\, A_t}, \tag{33}$$

$$\Omega_{tT} = \nu_t \left[\frac{b(T)}{P_{0T} + b(T)\, A_t} - \frac{b(t)}{P_{0t} + b(t)\, A_t} \right]. \tag{34}$$

Let us suppose that the Markov process generating the market filtration is

$$L_{tU} = \sigma\, t\, X_U + \beta_{tU}. \tag{35}$$

Then, the Brownian motion is given by

$$\mathrm{d}W_t = \mathrm{d}L_{tU} - \frac{1}{U-t} \left[\sigma\, U\, \mathbb{E}\left[X_U \,|\, L_{tU} \right] - L_{tU} \right] \mathrm{d}t, \tag{36}$$

and

$$\vartheta_t = \frac{\sigma U}{U-t} \mathbb{E}^{\mathbb{P}} \left[X_U \,|\, L_{tU} \right]. \tag{37}$$

Furthermore, in the case of the quadratic models, we have

$$\nu_t = \frac{2U}{(U-t)^2}\, L_{tU}. \tag{38}$$

In the case of exponential quadratic models, we have

$$\nu_t = \frac{\left(\frac{U-t}{U}\right)^{1/2}}{U-t} L_{tU} \exp\left[\frac{L_{tU}^2}{2(U-t)}\right]. \tag{39}$$

Dynamical equation under the risk-neutral measure $\mathbb{Q}$

We are now in the position to define (in a natural way) an equivalent risk-neutral measure $\mathbb{Q}$ by the following Radon-Nikodym derivative:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left[-\int_0^t \lambda_s \,\mathrm{d}W_s - \tfrac{1}{2}\int_0^t \lambda_s^2 \,\mathrm{d}s\right]. \tag{40}$$

By the Girsanov Theorem, we may define a $\mathbb{Q}$-Brownian motion $\{W_t^{\mathbb{Q}}\}_{0\le t<U}$ by

$$\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t + \lambda_t \,\mathrm{d}t. \tag{41}$$

The dynamical equation of the bond price has thus the following $\mathbb{Q}$-form:

$$\mathrm{d}P_{tT} = r_t \, P_{tT} \,\mathrm{d}t + \Omega_{tT} \, P_{tT} \,\mathrm{d}W_t^{\mathbb{Q}}. \tag{42}$$

Furthermore, the discount bond price at time t is given by the familiar risk-neutral valuation formula:

$$P_{tT} = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\,\middle|\,L_{tU}\right], \tag{43}$$

where

$$r_t = -\frac{\partial_t P_{0t} + A_t\,\partial_t b(t)}{P_{0t} + b(t)A_t}. \tag{44}$$

The dynamical equation for the short rate of interest $\{r_t\}$ is given by

$$\frac{\mathrm{d}r_t}{r_t} = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t^{\mathbb{Q}}, \tag{45}$$

where

$$\mu_t = \frac{\partial_t P_{0t} + A_t\,\partial_t b(t)}{P_{0t} + b(t)\,A_t} - \frac{\partial_{tt} P_{0t} + A_t\,\partial_{tt} b(t)}{\partial_t P_{0t} + A_t\,\partial_t b(t)}, \tag{46}$$

and

$$\sigma_t = \nu_t\left(\frac{b(t)}{P_{0t} + b(t)\,A_t} - \frac{\partial_t b(t)}{\partial_t P_{0t} + A_t\,\partial_t b(t)}\right). \tag{47}$$

Multi-factor models

So far, the considered pricing kernel models have mainly been driven by a single stochastic factor.

The presented asset pricing framework is flexible, tractable, and intuitive so that the extension to the multi-factor case is rather natural.

As an example, let the market filtration $\{\mathcal{F}_t\}$ be jointly generated by two Markov processes $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$.

We consider the following bond price model:

$$P_{tT} = \frac{f_0(T) + f_1(T)\, Y_{tT}^{(1)} + f_2(T)\, Y_{tT}^{(2)}}{f_0(t) + f_1(t)\, Y_{tt}^{(1)} + f_2(t)\, Y_{tt}^{(2)}}, \tag{48}$$

where, for $i = 1, 2$,

$$Y_{tT}^{(i)} = \int_{T-t}^{U-t} \mathbb{E}\left[F_i(t+u, X_{t+u}^{(i)}) \,|\, X_t^{(1)}, X_t^{(2)}\right] w_i(T, u-T+t)\,\mathrm{d}u, \tag{49}$$

and where $f_0(t)$, $f_1(t)$, $f_2(t)$ are deterministic positive decreasing functions.

The associated pricing kernel process is

$$\pi_t = f_0(t) + f_1(t)\, Y^{(1)}_{tt} + f_2(t)\, Y^{(2)}_{tt}. \tag{50}$$

Let us suppose that the driving factors are two independent "Brownian bridge information processes" $\{L^{(i)}_{tU}\}$, $i = 1, 2$.

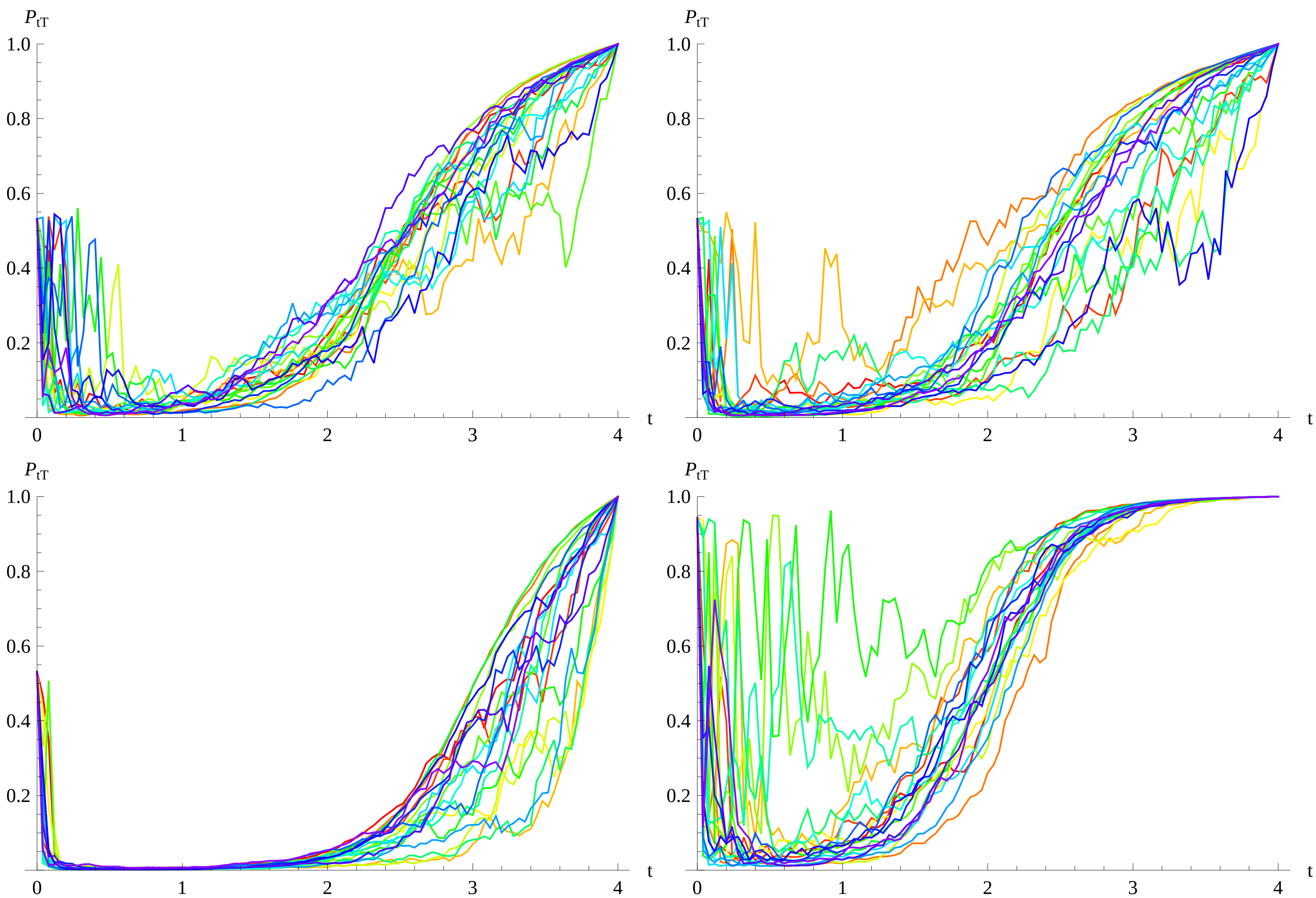
We may further assume that $Y^{(1)}_{tT}$ generates a quadratic model, and that $Y^{(2)}_{tT}$ gives rise to an exponential quadratic model.

The following simulations of the two-factor bond price models show different behaviours when compared to the one-factor models.

In the simulations below, the functions $f_0(t)$, $f_1(t)$, $f_2(t)$ decrease exponentially. The horizon is at $U = 6$, and the bond maturity is $T = 4$.

The various price dynamics are due to different values of the exponential decay parameter, and due to various values of σ_i in

$$L^{(i)}_{tU} = \sigma_i\, t\, X^{(i)}_U + \beta^{(i)}_{tU}. \tag{51}$$


P_tT
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
t

Incomplete market models driven by LRBs

So far, we have generated the market filtration by the Markov process

$$L_{tU} = \sigma\, t\, X_U + \beta_{tU}. \tag{52}$$

This process belongs to the class of time-inhomogeneous Markov processes we call "Lévy random bridge" (LRB).

Definition (Multivariate LRB). $\{L_{tU}\}_{0\leq t\leq U}$ is a multivariate LRB on $\mathbb{R}^m$ if:

1. The random variable L_{UU} on $\mathbb{R}^m$ has marginal law ν.
2. There exist a multivariate Lévy process $\{L_t\}_{0\leq t\leq U}$ on $\mathbb{R}^m$ such that L_t has multivariate density function $\rho_t(x)$ on $\mathbb{R}^m$ for all $t \in (0, U]$.
3. The marginal law ν concentrates mass where $\rho_U(z)$ is positive and finite, that is $0 < \rho_U(z) < \infty$ for ν-almost-every $z \in \mathbb{R}^m$.
4. For every $n \in \mathbb{N}_+$, every $0 < t_1 < \ldots < t_n < U$, every $(x_1, \ldots, x_n) \in \mathbb{R}^m \times \mathbb{R}^n$, and ν-almost-every $z \in \mathbb{R}^m$, we have

$$\mathbb{P}\left[L_{t_1U} \leq x_1, \ldots, L_{t_nU} \leq x_n \,\middle|\, L_{UU} = z\right] = \mathbb{P}\left[L_{t_1} \leq x_1, \ldots L_{t_n} \leq x_n \,\middle|\, L_U = z\right].$$

Proposition ("Lévy probability measure" $\mathbb{L}$)

Let $\{L_{tU}\}_{0\leq t\leq U}$ denote a multivariate LRB with marginal law ν. Let the multivariate Lévy process $\{L_t\}_{0\leq t\leq U}$, which generates the LRB, have density $\rho_t(x)$ for all $t\in(0,U]$. Under the measure $\mathbb{L}$ defined by

$$\ell_t^{-1} := \left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{L}}\right|_{\mathcal{F}_t} = \int_{\mathbb{R}} \frac{\rho_{U-t}(z-L_{tU})}{\rho_U(z)}\,\nu(\mathrm{d}z), \tag{53}$$

the LRB $\{L_{tU}\}$ has the law of the generating Lévy process for $t\in[0,U)$.

Proposition ($\mathbb{L}$-independence)

Let $\{L_{tU}\}_{0\leq t\leq U}$ be a multivariate LRB on $\mathbb{R}^m$ generated by a Lévy process on $\mathbb{R}^m$ of which multivariate density function $\rho_t(z)$ factorises, that is

$$\rho_t(z) = \prod_{k=1}^{m} \rho_t^k(z_k). \tag{54}$$

Under $\mathbb{L}$, the components, $\{L_{tU}^{(i)}\}$ and $\{L_{tU}^{(j)}\}$ $i\neq j$, of the LRB on $\mathbb{R}^m$ are independent and each LRB component has the law of the respective component of the generating Lévy process $\{L_t\}$, for $t\in[0,U)$.

Spiralling debt and global bond markets

We shift the emphasis on the value of a sovereign bond reflecting the level of economic health of the issuing country.

We consider a central government with

(i) a source of income (tax revenues, state-owned companies) and
(ii) ordinary expenditures (health care, public education, armed forces).

We model the combination of income and ordinary expenditures by a Brownian random bridge:

$$L_{tU}^{(1)} = \sigma\, t\, X_U^{(1)} + \beta_{tU}.$$

In addition, we assume that there is an accumulation of extraordinary losses (natural catastrophes, war, financial crises with bank bailouts) resulting in spiralling debt.

We model the spiralling debt by a gamma random bridge:

$$L_{tU}^{(2)} = X_U^{(2)}\, \gamma_{tU},$$

where $\{\gamma_{tU}\}$ is an independent standard gamma bridge over $[0, U]$.

The random variables $X_U^{(1)}$ and $X_U^{(2)}$ may be dependent, though.

Next, we assume that the pricing kernel model,

$$\pi_t = f_0(t) + f_1(t) \int_0^{U-t} \mathbb{E}^{\mathbb{L}} \left[F(t+u, L_{t+u,U}) \,|\, L_{tU} \right] w(t,u) \mathrm{d}u, \tag{55}$$

for the considered economy is constructed by selecting functions of the form

$$F(t+u_1, t+u_2, y_1+x_1, y_2+x_2) = \exp\left(-a(y_1+x_1) + c(y_2+x_2)\right),$$

$$w(t, u_1, u_2) = \exp\left(-\frac{a^2}{2}(t+u_1)\right)(1-c)^{m(t+u_2)}.$$

Here, $L_{tU} = \left(L_{tU}^{(1)}, L_{tU}^{(2)}\right)$ and $u = (u_1, u_2)$.

It follows that the bond price process is given by:

$$P_{tT} = \frac{P_{0T} + b(T)\, A_t^{\mathbb{L}}}{P_{0t} + b(t)\, A_t^{\mathbb{L}}}, \tag{56}$$

where, for $0 \leq t \leq T < U$, $a \geq 0$, $0 \leq c \leq 1$, $m > 0$, we have

$$b(t) = \frac{(U-t)^2 f_1(t)}{1 + f_1(0)U^2}, \quad A_t^{\mathbb{L}} = (1-c)^{mt} \exp\left(-a\, L_{tU}^{(1)} - \tfrac{1}{2}a^2 t + c\, L_{tU}^{(2)}\right) - 1.$$

Dependence in international markets

The effects arising from spiralling debt affect international markets, and the deterioration of an economy's health exposes, for instance, foreign creditors holding debt of the distressed economy.

Bond markets are global "debt networks" linking several national economies to one another. The result of such networks is usually contagion.

An ailing economy may severely damage creditors which, through financial exposure, may suffer losses due to contagion effects.

In the next situation, we consider a network of four Eurozone countries, which are linked according to the following exposure matrix:

	GER	FRA	ESP	ITA
GER	1	0	0	0
FRA	0	1	0	0
ESP	0.57	0.47	1	0
ITA	0.49	0.25	0	1

By use of the incomplete market approach driven by LRBs, we are in the position to model contagion effects with ease.

We introduce a linear combination $\{\widetilde{L}^{(j)}_{tU}\}$ of cumulative random bridge processes to model the exposure of country j to the other network economies i with cumulating debt $\{L^{(i)}_{tU}\}$:

$$\widetilde{L}^{(j)}_{tU} = \sum_{i}^{n} w^{(j)}_i \, L^{(i)}_{tU}.$$

The processes $\{L^{(i)}_{tU}\}_{i=1,\dots,n}$ may be dependent via their joint terminal law.

A similar calculation as for the single-economy example leads to economy's j bond price process with form:

$$P_{tT} = \frac{P_{0T} + b(T)\, A^{\mathbb{L}}_t}{P_{0t} + b(t)\, A^{\mathbb{L}}_t}, \tag{57}$$

where

$$\begin{aligned} b(t) &= \frac{(U-t)^{n+1} f_1(t)}{1 + f_1(0) U^{n+1}}, \\ A^{\mathbb{L}}_t &= \prod_{i=1}^{n} \left(1 - w^{(j)}_i\right)^{m_i t} \exp\left(-a\, L^{(j)}_{tU} - \tfrac{1}{2}a^2 t + \widetilde{L}^{(j)}_{tU}\right) - 1, \end{aligned} \tag{58}$$

and $0 \leq w_i^{(j)} \leq 1$, $m_i > 0$.

It is straightforward to simulate the bond price process, and so it is for the yield and the spread process:

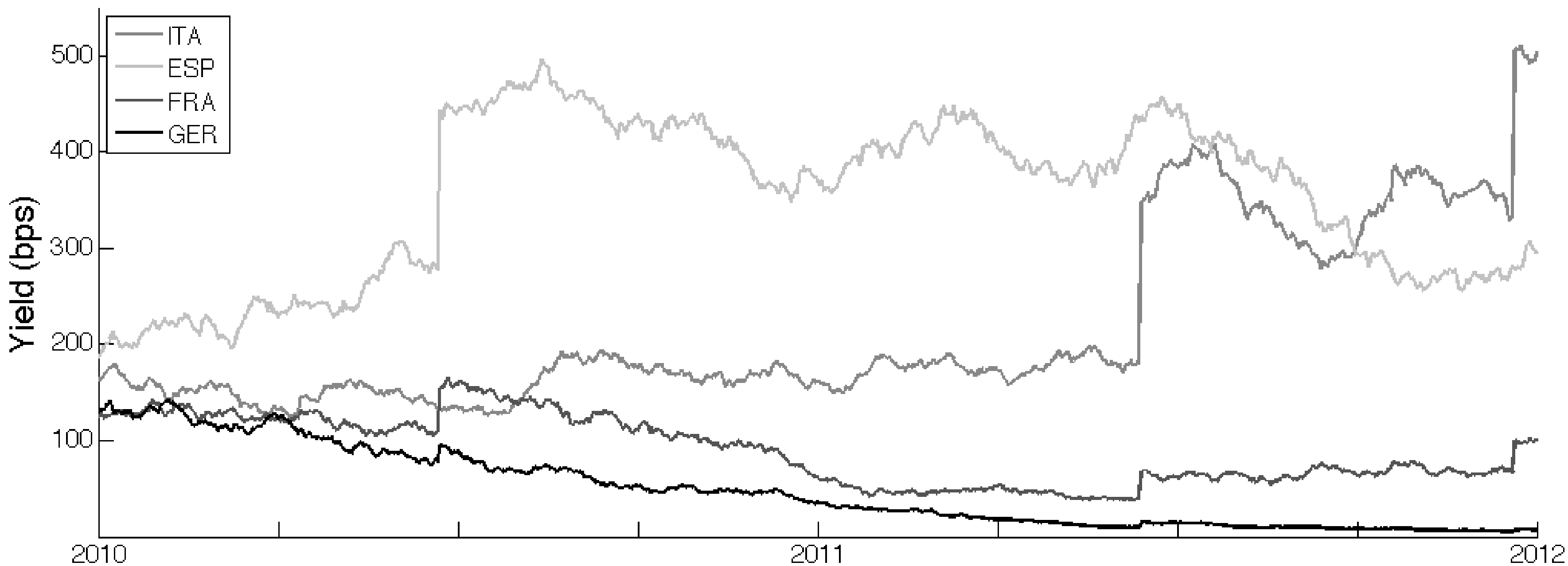


Figure 1: Simulation of the yield process of the two-year-maturity bonds issued by Germany, France, Italy, and Spain.

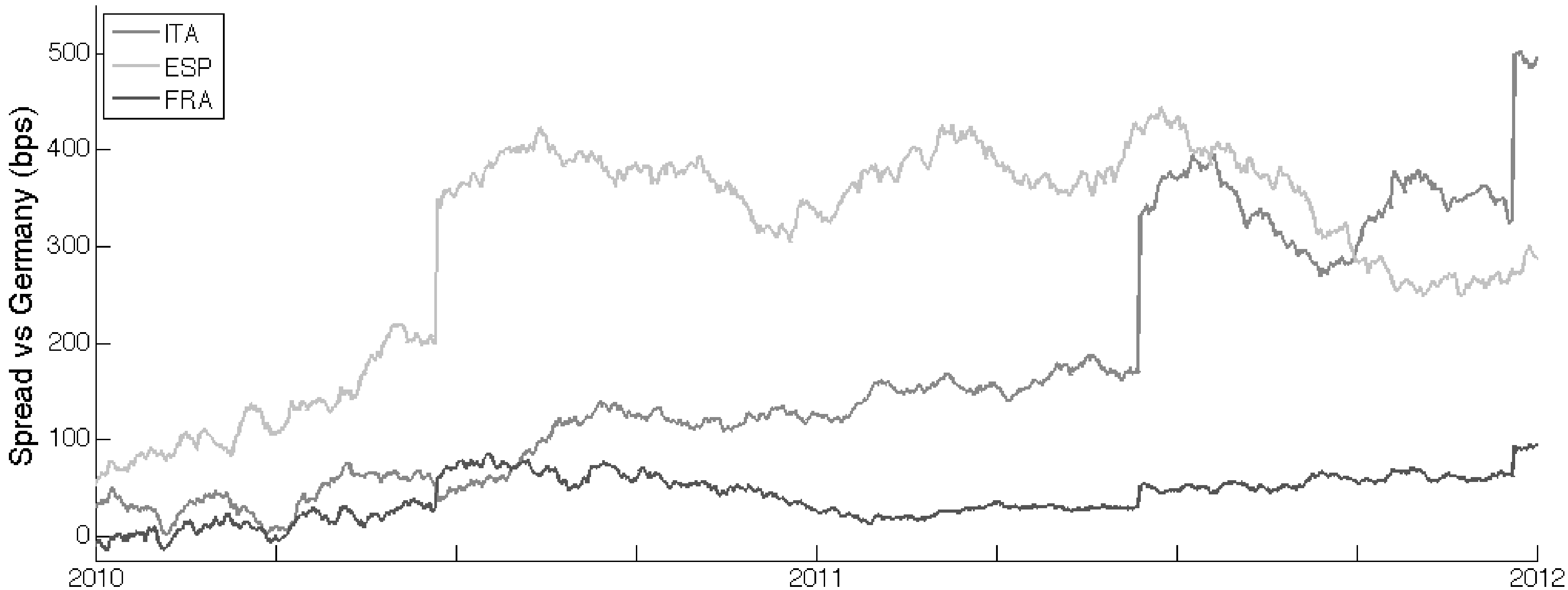


Figure 2: Simulation of the price spread process for the two-year-maturity bonds issued by France, Italy, and Spain compared with the two-year bond issued by Germany.

General asset pricing

The presented asset pricing framework can be applied to price any financial asset in a way that is consistent with the valuation of fixed-income securities.

We may consider a non-dividend-paying stock (e.g., Google) with a random value S_{TT} at a future fixed date T, where $0 \leq t \leq T < U$.

"No-arbitrage" requires that the product of an asset price process $\{S_{tT}\}$ with the pricing kernel $\{\pi_t\}$ be a $\mathbb{P}$-martingale. That is:

$$\pi_t \, S_{tT} = m_{tT}, \tag{59}$$

where $\{m_{tT}\}$ are families of $\mathbb{P}$-martingales.

For example,

$$m_{tT} = g_0(T) + g_1(T) Z_{tT} \tag{60}$$

where $g_0(T)$ and $g_1(T)$ are deterministic, and $\{Z_{tT}\}$ is defined by

$$Z_{tT} = \int_{T-t}^{U-t} \mathbb{E}^{\mathbb{P}} \left[G\left(t+u, X_{t+u}\right) \mid X_t \right] \psi(t+u) \, \mathrm{d}u. \tag{61}$$

The function $\psi(t)$ is deterministic, and $G(t,x)$ is taken to be measurable.

Assuming that the pricing kernel is of the form

$$\pi_t = f_0(t) + f_1(t) Y_{tt}, \tag{62}$$

it follows that the asset price process $\{S_{tT}\}$ is given by

$$S_{tT} = \frac{S_{0T} + z(T)\left(Z_{tT} - Z_{0T}\right)}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}, \tag{63}$$

where $y(t)$ and $z(t)$ are deterministic functions.

Proposition (Asset price model)

Let $\{M_t\}_{0\leq t<U}$ be the $\{\mathcal{F}_t\}$-adapted density martingale inducing a change-of-measure from $\mathbb{P}$ to $\mathbb{M}$. Let $\{A^{(i)}_t\}^{i=1,2}_{0\leq t<U}$ be $(\{\mathcal{F}_t\},\mathbb{M})$-martingales, and consider a pricing kernel model of the form

$$\pi_t = \pi_0 \left[P_{0t} + b_2(t) A^{(2)}_t \right] M_t. \tag{64}$$

Furthermore, let

$$S_{TT} = \frac{S_{0T} + b_1(T)\, A^{(1)}_T}{P_{0T} + b_2(T)\, A^{(2)}_T},$$

where $b_i(T)$, $i=1,2$, are deterministic functions, and $b_2(t)$ is non-negative and non-increasing. Then the associated asset price process $\{S_{tT}\}$ is given by

$$S_{tT} = \frac{S_{0T} + b_1(T)\, A^{(1)}_t}{P_{0t} + b_2(t)\, A^{(2)}_t}. \tag{65}$$

One example is...

$$S_{tT} = \frac{S_{0T} + \frac{(U-T)^{\eta-1/2}U^{1/2}g_1(T)}{1+g_1(0)U^\eta}\left[\sqrt{1-t/U}\ \exp\left(\frac{L_{tU}^{(1)\,2}}{2(U-t)}\right)-1\right]}{P_{0t} + \frac{(U-t)^4 f_1(t)}{4U\left[1+(1/12)\,f_1(0)U^3\right]}\left[\frac{U}{(U-t)^2}\,L_{tU}^{(2)\,2} - \frac{t}{U-t}\right]}. \tag{66}$$

In the case that the Markov process $\{L_{tU}\}$ is multidimensional, that is, $L_{tU} = (L_{tU}^{(1)}, L_{tU}^{(2)}, \dots, L_{tU}^{(n)})$, then the asset with price process (66) is traded in an incomplete market.

Proposition (Dynamical equation, $\mathbb{P}$ to $\mathbb{Q}$)

Let $\{\mathcal{F}_t\}$ be jointly generated by $\{L_{tU}^{(i)}\}$, $i=1,2$. Let the price process of an asset be of the form (65) where $\{A_t^{(i)}\}_{0\le t<U}$ satisfies

$$\mathrm{d}A_t^{(i)} = \nu_t^{(i)}\left(\mathrm{d}W_t^{\mathbb{P}\ (i)} + \vartheta_t^{(i)}\mathrm{d}t\right)$$

for $i=1,2$. The process $\{\nu_t^{(i)}\}$ is $\{\mathcal{F}_t\}$-adapted, and

$$\vartheta_t^{(i)} = \frac{\sigma_i\,U}{U-t}\,\mathbb{E}^{\mathbb{P}}\left[X_U^{(i)}\,\middle|\,\mathcal{F}_t\right],$$

$$\mathrm{d}W_t^{\mathbb{P}\ (i)} = \mathrm{d}L_{tU}^{(i)} - \frac{1}{U-t}\left(\sigma_i\,U\,\mathbb{E}^{\mathbb{P}}\left[X_U^{(i)}\,\middle|\,\mathcal{F}_t\right] - L_{tU}^{(i)}\right)\mathrm{d}t$$

where σ_i is constant. The dynamical equation of such a price process is

$$\frac{\mathrm{d}S_{tT}}{S_{tT}} = (r_t + \lambda_t \Sigma_{tT})\,\mathrm{d}t + \Sigma_{tT}\,\mathrm{d}W_t^{\mathbb{P}}, \tag{67}$$

where

$$r_t = -\frac{\partial_t P_{0t} + A_t^{(2)}\,\partial_t b_2(t)}{P_{0t} + b_2(t)\,A_t^{(2)}}, \qquad \lambda_t = \begin{pmatrix} \vartheta_t^{(1)} - \rho_{ij}\,\nu_t^{(2)}\,\frac{b_2(t)}{P_{0t}+b_2(t)\,A_t^{(2)}} \\ \\ \vartheta_t^{(2)} - \nu_t^{(2)}\,\frac{b_2(t)}{P_{0t}+b_2(t)\,A_t^{(2)}} \end{pmatrix},$$

$$\Sigma_{tT} = \begin{pmatrix} \frac{b_1(T)\nu_t^{(1)}}{S_{0T}+b_1(T)\,A_t^{(1)}} \\ \\ -\frac{b_2(t)\nu_t^{(2)}}{P_{0t}+b_2(t)\,A_t^{(2)}} \end{pmatrix}.$$

The process $W_t^{\mathbb{P}} = (W_t^{\mathbb{P}\,(1)}, W_t^{\mathbb{P}\,(2)})$ is a two-dimensional $(\{\mathcal{F}_t\},\mathbb{P})$-Brownian motion where $\mathrm{d}W_t^{\mathbb{P}\,(i)}\,\mathrm{d}W_t^{\mathbb{P}\,(j)} = \rho_{ij}\,\mathrm{d}t$ for $i \neq j$, $\rho_{ij} \in [-1,1)$, and $\mathrm{d}W_t^{\mathbb{P}\,(i)}\,\mathrm{d}W_t^{\mathbb{P}\,(j)} = \mathrm{d}t$ for $i = j$.

We can now write the dynamical equation of the asset price process under the risk-neutral measure $\mathbb{Q}$:

$$\frac{\mathrm{d}S_{tT}}{S_{tT}} = r_t\,\mathrm{d}t + \Sigma_{tT}\,\mathrm{d}W_t^{\mathbb{Q}}, \tag{68}$$

where $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t^{\mathbb{P}} + \lambda_t \mathrm{d}t$ is the risk-neutral Brownian motion defined in terms of the $\mathbb{P}$-Brownian motion $\{W_t^{\mathbb{P}}\}$ and the market price of risk process $\{\lambda_t\}$.

The solution to the stochastic differential equation (68) has the familiar $\mathbb{Q}$-log-normal form

$$S_{tT} = S_{0T} \exp\left(\int_0^t \left(r_s - \tfrac{1}{2}\,\Sigma_{sT}^2\right) \mathrm{d}s + \int_0^t \Sigma_{sT}\,\mathrm{d}W_s^{\mathbb{Q}} \right). \tag{69}$$

Thus we can construct multi-factor asset price processes (e.g. equity) with stochastic discounting and stochastic volatility, which are consistent with the dynamics under the real measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$.

Inflation-linked assets

The pricing kernel approach to asset pricing can naturally be applied to the pricing of inflation-linked securities.

The idea here is to introduce a pricing kernel $\{\pi_t\}$ for the nominal economy and a pricing kernel $\{\pi_t^R\}$ for the real economy.

The price index process $\{C_t\}$ (CPI, RPI) is then modelled as an exchange rate between the two economies, that is

$$C_t = \frac{\pi_t^R}{\pi_t}. \tag{70}$$

The price at time $t \leq T$ of an inflation-linked discount bond P_{tT}^{IL} that pays C_T at bond maturity T is

$$P_{tT}^{IL} = \frac{1}{\pi_t}\mathbb{E}\left[\pi_T\, C_T \,|\mathcal{F}_t\right] \tag{71}$$

$$= \frac{1}{\pi_t}\mathbb{E}\left[\pi_T^R \,|\mathcal{F}_t\right]. \tag{72}$$

Next, we consider:

$$\pi_t = f_0(t) + f_1(t)\int_0^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) \,|\, X_t\right] w(t,u)\mathrm{d}u, \tag{73}$$

$$\pi_t^R = f_0^*(t) + f_1^*(t)\int_0^{U-t} \mathbb{E}\left[F^*(t+u, X_{t+u}) \,|\, X_t\right] w^*(t,u)\mathrm{d}u, \tag{74}$$

where the real pricing kernel process $\{\pi_t^R\}$ no longer must be a supermartingale, and the Markov process $\{X_t\}_{0\leq t\leq U}$ is multi-dimensional.

By inserting the expressions for the pricing kernels in the pricing formula for the inflation-linked discount bond, we obtain

$$P_{tT}^{IL} = \frac{P_{0T}^{IL} + y^*(T)\left(Y_{tT}^* - Y_{0T}^*\right)}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}, \tag{75}$$

where $\{Y_{tT}\}$ and $y(t)$ are defined by (10) and (11), and

$$Y_{tT}^* = \int_{T-t}^{U-t} \mathbb{E}\left[F^*(t+u, X_{t+u}) \,|\, X_t\right] w^*(T, u-T+t)\mathrm{d}u, \tag{76}$$

$$y^*(t) = \frac{f_1^*(t)}{1 + f_1(0)Y_{00}}. \tag{77}$$

For the pricing of inflation-linked securities, one may wish to consider models with a higher degree of tractability which yield inflation-adjusted discount bond price processes of the form

$$P_{tT}^{IL} = \frac{P_{0T}^{IL} + b^*(T)\, A_t^*}{P_{0t} + b(t)\, A_t}, \tag{78}$$

where $\{A_t\}$ and $\{A_t^*\}$ are martingales with respect to an auxiliary martingale measure $\mathbb{M}$.

One can now proceed to price inflation-linked derivatives such as inflation-linked swaps and swaptions.

Conclusions

1. The presented pricing models, derived under $\mathbb{P}$, are flexible and exhibit a great deal of flexibility. Model calibration is in part automatic.

2. Bonds and equity are priced coherently. Other asset classes such as FX, inflation-linked securities, credit-risky and insurance products are next in the research agenda.

3. The proposed framework gives rise to novel stochastic positive interest rate and stochastic volatility models.

4. The structure of the constructed price processes offers a natural base for dependence modelling and thus for the pricing of asset portfolios.

5. Future work may include the study of the volatility surface models related to this pricing framework, and the inclusion of regulatory requirements via the degrees of freedom inherent in the pricing models.

Some related recent work

J. Akahori, Y. Hishida, J. Teichmann, T. Tsuchiya (2009) A Heat Kernel Approach to Interest Rate Models, arXiv.org, No. 0910.5033

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