Linear-Rational Term Structure Models

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Goals

- Three desirable feature of a term structure model:
  - Tractable pricing formulas (for zero-coupon bonds this is a necessity, but clearly desirable also for more complicated contracts such as swaptions)
  - Nonnegative short rate
  - Unspanned Stochastic Volatility

- Affine term structure models have great difficulty combining these features

- **Goal:** Develop a framework where all these features are naturally present

- Illustrate on swaption pricing

Outline

- Linear-Rational Term Structure Models
- Unspanned Stochastic Volatility
- Swaption Pricing
- Empirics
- Conclusion

Linear-Rational

Term Structure Models

State price density models

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- Model price at t of any claim C maturing at T:

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- Relation to short rate r_t and pricing measure $\mathbb{Q}$:

$$\zeta_t \quad \propto \quad \mathrm{e}^{-\int_0^t r_s ds}\,\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \;\middle|\; \mathcal{F}_t\right]$$

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This approach was used by

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- Rogers (1997)
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How to tractably model ζ_t?

Linear-Rational term structure models

Ingredients:

- Factor process X with state space $E \subset \mathbb{R}^d$
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Key idea (Linear-Rational Term Structure model):

- $p_\zeta(x) = \phi + \psi^\top x$, positive on E
- X with affine drift:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \mathrm{d}M_t,$$

where $\kappa \in \mathbb{R}^{d \times d}$, $\theta \in \mathbb{R}^d$, M is a martingale.

Linear-Rational term structure models

Lemma. *The conditional expectation of X_T is*

$$\mathbb{E}\left[X_T \mid \mathcal{F}_t\right] = \theta + \mathrm{e}^{-\kappa(T-t)}(X_t - \theta)$$

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Consequences:

- **Linear-rational** (and explicit) bond price system:

$$P(t, t+\tau) = \frac{\mathrm{e}^{-\alpha\tau}}{p_\zeta(X_t)}\mathbb{E}[p_\zeta(X_{t+\tau}) \mid \mathcal{F}_t] = F(\tau, X_t),$$

where $F(\tau, x) = \dfrac{(\phi + \psi^\top\theta)\mathrm{e}^{-\alpha\tau} + \psi^\top\mathrm{e}^{-(\alpha+\kappa)\tau}(x - \theta)}{\phi + \psi^\top x}$

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- **Linear-rational** short rate: $r_t = \alpha - \dfrac{\psi^\top \kappa\,(\theta - X_t)}{\phi + \psi^\top X_t}$

Intrinsic choice of α

Define

$$\alpha^* = \sup_{x \in E} \frac{\psi^\top \kappa (\theta - x)}{\phi + \psi^\top x} \qquad \alpha_* = \inf_{x \in E} \frac{\psi^\top \kappa (\theta - x)}{\phi + \psi^\top x}.$$

- Should arrange so that $\alpha^* < \infty$ to get r_t bounded below
- With $\alpha = \alpha^*$, we get

$$r_t \in [0, \alpha^* - \alpha_*]$$

- For the model to be useful, this range must be wide enough

Unspanned Stochastic Volatility

Unspanned stochastic volatility in Linear-rational models

Empirical fact: Volatility risk cannot be hedged using bonds

- Collin-Dufresne & Goldstein (02): Interest rate swaps can hedge only 10%–50% of variation in ATM straddles (a volatility-sensitive instrument)

- Heidari & Wu (03): Level/curve/slope explain 99.5% of yield curve variation, but 59.5% of variation in swaption implied vol

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Definition. *The state process has* **unspanned factors** *if the current state* X_t *cannot be inferred from* $\{P(t, t+\tau), \tau \geq 0\}$*. Equivalently, the map* $E \ni x \mapsto F(\cdot, x)$ *is not injective.*

Unspanned stochastic volatility in Linear-rational models

Theorem. *Assume that* $\mathrm{int}(E) \neq \emptyset$ *and that all eigenvalues of* κ *are nonzero. The following are equivalent:*

(i) *The state process has unspanned factors*

(ii) *There exists* $u \in \mathbb{R}^d \setminus \{0\}$ *such that* $F(\cdot, x) \equiv F(\cdot, x + su)$ *for all* $x \in \mathbb{R}^d$ *and all* $s \in \mathbb{R}$

(iii) *There exists* $u \in \mathbb{R}^d \setminus \{0\}$ *such that* $\psi^\top \mathrm{e}^{-\kappa\tau} u = 0$*, all* $\tau \geq 0$

Any u that works in (ii) also works in (iii), and vice versa.

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Define the subspace U of unspanned directions:

$$U = \left\{ u \in \mathbb{R}^d : \psi^\top \mathrm{e}^{-\kappa\tau} u = 0 \text{ for all } \tau \geq 0 \right\}$$

The "number of unspanned factors" is the dimension of U.

Unspanned stochastic volatility in Linear-rational models

When do we have unspanned factors?

Theorem. *Let $\lambda_1, \dots, \lambda_n$ ($n \leq d$) denote the distinct eigenvalues of κ, and let $m_1, \dots, m_n$ be their geometric multiplicities. Then*

$$\dim U \geq (m_1 - 1) + \cdots + (m_n - 1).$$

If κ is diagonalizable with real eigenvalues, and ψ is not orthogonal to any eigenspace $\mathrm{Ker}(\lambda_i - \kappa)$, $i = 1, \dots, n$, the above inequality is in fact an equality.

Constructing models with USV

By previous theorem, need geometric multiplicity of eigenvalues of κ. We can do this by adding factors to an initial model.

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  with $\widehat{\kappa}$ unrestricted. Suppose this can capture the dynamics of the yield curve (in practice, $d = 3$ is enough.)

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- "Generically" (on a full-measure set of parameters), no unspanned factors are present.
- Suppose want to include swaptions; need unspanned factors.
- Idea: Construct a $(d + k)$-factor model that is observationally equivalent to a d-factor model when calibrated to bonds only.

Constructing models with USV

Consider now a $(d+k)$-factor model on $E \subset \mathbb{R}^{d+k}$ of the form:

$$\mathrm{d}X_t = \kappa\,(\theta - X_t)\,\mathrm{d}t + \mathrm{d}M_t, \qquad p_\zeta(x) = \phi + \psi^\top x.$$

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Theorem. *Let $A : \mathbb{R}^{d+k} \to \mathbb{R}^d$ be linear and define $\widehat{X} = AX$. Then*

$$\mathrm{d}\widehat{X}_t = \widehat{\kappa}\,(\widehat{\theta} - \widehat{X}_t)\mathrm{d}t + \mathrm{d}\widehat{M}_t, \qquad \widehat{M} = AM,$$

if and only if $A\kappa = \widehat{\kappa}A$ and $\widehat{\kappa}A\theta = \widehat{\kappa}\widehat{\theta}$.

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if and only if $A\kappa = \widehat{\kappa}A$ and $\widehat{\kappa}A\theta = \widehat{\kappa}\widehat{\theta}$.

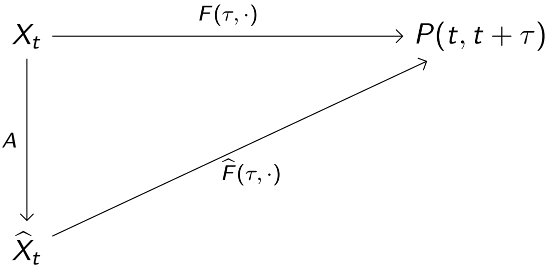
Furthermore, let $P(t,T)$ and $\widehat{P}(t,T)$ be the respective bond prices. Then

$$P(t,T) = \widehat{P}(t,T) \quad \textit{for all} \quad 0 \le t \le T$$

if and only if $\widehat{\phi} = \phi$ and $A^\top \widehat{\psi} = \psi$.

Constructing models with USV

The extended model (X, p_ζ) has unspanned factors:



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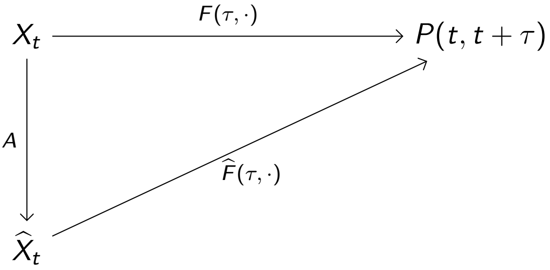

Hence for $u \in \mathrm{Ker}(A)$ we have

$$F(\tau, x + su) = F(\tau, x) \quad \text{for all} \quad \tau \geq 0,\ s \in \mathbb{R}.$$

Therefore, $\dim U \ \geq \ \dim \mathrm{Ker}(A) \ \geq \ k$.

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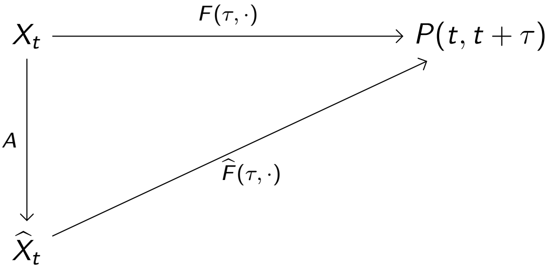

Task: Find some A and a class of κ and $\widehat{\kappa}$ such that $A\kappa = \widehat{\kappa}A$.
Any choice of θ, M then gives $\widehat{X}$ by setting

$$\widehat{\theta} = A\theta, \quad \widehat{M} = AM.$$

Given $\widehat{\phi}$, $\widehat{\psi}$ we get ϕ, ψ by setting $\phi = \widehat{\phi}$, $\psi = A^\top \widehat{\psi}$.

Constructing models with USV

Example ($d = 3$, $k = 1$, first factor unspanned): Set

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \end{pmatrix} = AX = \begin{pmatrix} X_1 + X_4 \\ X_2 \\ X_3 \end{pmatrix}$$

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Define

$$\kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \\ \kappa_{21} & \kappa_{22} & \kappa_{21} & \kappa_{21} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{31} \\ & & & \kappa_{11} \end{pmatrix}, \qquad \widehat{\kappa} = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{21} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix}$$

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Then $A\kappa = \widehat{\kappa}A$, and $\dim U = 1$ for generic parameter values.

Note: κ only depends on $3 \times 3 = 9$ parameters.

Constructing models with USV

Example ($d = 3$, $k = 2$, first and second factors unspanned):

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \end{pmatrix} = AX = \begin{pmatrix} X_1 + X_4 \\ X_2 + X_5 \\ X_3 \end{pmatrix}$$

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Then $A\kappa = \widehat{\kappa}A$, and $\dim U = 2$ for generic parameter values.

Canonical representation

Theorem. *Assume* $E = \mathbb{R}^d_+$, *consider any linear-rational model with interest rates bounded below. Then, w.l.o.g. one can take*

$$p_\zeta(x) = 1 + \mathbf{1}_m^\top x,$$

where $\mathbf{1}_m = (\underbrace{1, \ldots, 1}_{m \text{ times}}, 0, \ldots, 0) \in \mathbb{R}^d$.

The intrinsic choice $\alpha = \alpha^*$ *yields* $r_t \in [0, \alpha^* - \alpha_*]$, *where*

$$\alpha^* = \max\left\{\mathbf{1}_m^\top \kappa\theta,\ -\mathbf{1}_m^\top \kappa_1, \ldots,\ -\mathbf{1}_m^\top \kappa_d\right\}$$

$$\alpha_* = \min\left\{\mathbf{1}_m^\top \kappa\theta,\ -\mathbf{1}_m^\top \kappa_1, \ldots,\ -\mathbf{1}_m^\top \kappa_d\right\}$$

Swaption pricing

Interest rate swaps

- Exchange a stream of fixed-rate for floating-rate payments
- Consider a tenor structure,

$$T_0 < T_1 < \cdots < T_n, \qquad \Delta = T_i - T_{i-1} \text{ fixed.}$$

- Pre-determined swap rate K. At T_i, $1 \leq i \leq n$,
  - pay ΔK,
  - receive LIBOR, $\Delta L(T_{i-1}, T_i) = \Delta \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right)$.

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  - receive LIBOR, $\Delta L(T_{i-1}, T_i) = \Delta \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right)$.
- Value of swap at $t \le T_0$:

$$\Pi_t^{\text{swap}} = \underbrace{P(t, T_0) - P(t, T_n)}_{\text{floating leg}} - \underbrace{\Delta K \sum_{i=1}^{n} P(t, T_i)}_{\text{fixed leg}}$$

Swaptions

- Swaption = option to enter the swap at $T = T_0$
- The value at expiry T is

$$C_T = \left(\Pi_T^{\mathrm{swap}}\right)^+ = \left(\sum_{i=0}^{n} c_i P(T, T_i)\right)^+,$$

where $c_0 = 1$, $c_1 = \cdots = c_{n-1} = -\Delta K$, $c_n = -1 - \Delta K$.

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where $c_0 = 1$, $c_1 = \cdots = c_{n-1} = -\Delta K$, $c_n = -1 - \Delta K$.

- Hence its price at $t \le T$ is

$$\Pi_t^{\mathrm{swpt}} = \frac{1}{\zeta_t}\mathbb{E}\left[\zeta_T C_T \mid \mathcal{F}_t\right] = \frac{1}{\zeta_t}\mathbb{E}\left[p_{\mathrm{swap}}(X_T)^+ \mid \mathcal{F}_t\right],$$

where the **affine function** p_{swap} is given by

$$p_{\mathrm{swap}}(x) = \sum_{i=1}^{n} c_i \mathrm{e}^{-\alpha T_i}\mathbb{E}_x[p_\zeta(X_{T_i - T})]$$

Swaption pricing

- The swaption price is

$$\Pi_t^{\mathrm{swpt}} = \frac{1}{\zeta_t} \int_{\mathbb{R}^d} p_{\mathrm{swap}}(x)^+ F(\mathrm{d}x),$$

  where $F(\mathrm{d}x)$ is law of $(X_T \mid \mathcal{F}_t)$.

- For $d \geq 2$ this is numerically challenging

- Use Fourier techniques to reduce to line integral:

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- Use Fourier techniques to reduce to line integral:

Assume $\mathbb{E}[\mathrm{e}^{\mu p_{\mathrm{swap}}(X_T)}] < \infty$ for some $\mu > 0$. Then

$$\Pi_t^{\mathrm{swpt}} = \frac{1}{\zeta_t \pi} \int_0^\infty \mathrm{Re}\left[\frac{\widehat{q}(\mu + \mathrm{i}\lambda)}{(\mu + \mathrm{i}\lambda)^2} \right] \mathrm{d}\lambda$$

where $\widehat{q}(z) = \mathbb{E}\Big[\exp\Big(z\, p_{\mathrm{swap}}(X_T) \Big) \;\Big|\; \mathcal{F}_t \Big]$.

Empirics

Data

- Swap rates and implied ATM swaption (Bachelier) volatilities from Bloomberg
- Swap maturities T_n: 1Y, 2Y, 3Y, 5Y, 7Y, 10Y
- Swaptions: $T = 3$ month options on 1Y, 2Y, 3Y, 5Y, 7Y, 10Y (forward starting) swaps
- 827 weekly observations, Jan 29, 1997 — Nov 28, 2012
- Estimation approach: Quasi-maximum likelihood in conjunction with the (extended) Kalman filter

Calibration to swap rates

- 3-factor Linear-rational square-root (LRSQ) model:

$$\mathrm{d}\widehat{X}_t = \widehat{\kappa}(\widehat{\theta} - \widehat{X}_t)\mathrm{d}t \; + \; \mathrm{Diag}(\widehat{\sigma}_1\sqrt{\widehat{X}_{1t}}, \ldots, \widehat{\sigma}_3\sqrt{\widehat{X}_{3t}})\,\mathrm{d}\widehat{W}_t$$

$$\widehat{p}_\zeta(\widehat{x}) = 1 + \mathbf{1}^\top \widehat{x}$$

with $\widehat{\kappa}$ lower triangular for parsimony.

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- Results:

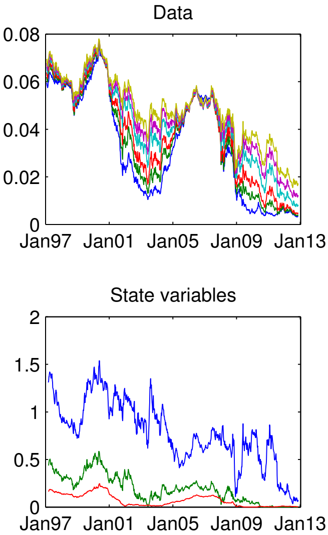
$$\widehat{\kappa} = \begin{pmatrix} 0.07 & 0 & 0 \\ -0.13 & 0.35 & 0 \\ 0.00 & -0.41 & 0.91 \end{pmatrix} \quad \widehat{\theta} = \begin{pmatrix} 0.97 \\ 0.36 \\ 0.16 \end{pmatrix} \quad \widehat{\sigma} = \begin{pmatrix} 0.40 \\ 0.33 \\ 0.10 \end{pmatrix}$$

- Range of short rates, $r_t \in [0, \alpha^* - \alpha_*]$:

$$\alpha^* - \alpha_* \approx 0.97$$

  Not a binding restriction.

Calibration to swap rates



Calibration to swap rates and swaptions

Two main challenges:

- Simultaneous fit to swaps and swaptions requires USV

  $\Longrightarrow$ introduce unspanned factors

- Efficient swaption pricing is necessary for calibration to time series data

Swaption pricing in the LRSQ model

- Recall swaption pricing formula:

$$\Pi^{\mathrm{swpt}} = \frac{\mathrm{e}^{\alpha t}}{p_\zeta(x)\pi} \int_0^\infty \mathrm{Re}\left[\frac{\widehat{q}(\mu + \mathrm{i}\lambda)}{(\mu + \mathrm{i}\lambda)^2}\right] \mathrm{d}\lambda$$

where $\widehat{q}(z) = \mathbb{E}_x\Big[\exp\Big(z\, p_{\mathrm{swap}}(X_T)\Big)\Big]$ with p_{swap} affine.

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- Exponential-affine transform formula: For any $u \in \mathbb{C}$, $v \in \mathbb{C}^d$,

$$\mathbb{E}_x\left[\mathrm{e}^{u + v^\top X_t}\right] = \mathrm{e}^{\Phi(t) + \Psi(t)^\top x}, \qquad x \in \mathbb{R}^d_+,$$

  where (Φ, Ψ) solves the Riccati system

$$\begin{cases} \Phi' = (\kappa\theta)^\top \Psi & \Phi(0) = u \\ \Psi_i' = -\kappa_i^\top \Psi + \frac{1}{2}\sigma_i^2 \Psi_i^2 & \Psi_i(0) = v_i, \quad i = 1, \dots, d \end{cases}$$

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where $\widehat{q}(z) = \mathbb{E}_x\Big[\exp\Big(z\, p_{\mathrm{swap}}(X_T)\Big)\Big]$ with p_{swap} affine.

- Currently, we can compute the prices at t_i, $i = 1, \ldots, 827$, of an ATM swaption in < 1 second in MATLAB on a standard desktop computer, with relative error $\approx 0.1\%$.

Calibration to swap rates and swaptions

Unspanned factors:

- State space $E = \mathbb{R}_+^{3+k}$,

$$\mathrm{d}X_t = \kappa\,(\theta - X_t)\mathrm{d}t + \mathrm{Diag}\Big(\sigma_1\sqrt{X_{1t}}, \ldots, \sigma_{3+k}\sqrt{X_{3+k,t}}\Big)\mathrm{d}W_t$$

where $\kappa \in \mathbb{R}^{(3+k)\times(3+k)}$, $\theta \in \mathbb{R}_+^{3+k}$, $\sigma_i > 0$ $(i = 1, \ldots, 3+k)$

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- If $k = 1$ we can take the first factor unspanned:

$$\kappa = \left(\begin{array}{ccc|c} \kappa_{11} & & & \\ \kappa_{21} & \kappa_{22} & & \kappa_{21} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{31} \\ \hline & & & \kappa_{11} \end{array}\right)$$

$\rightarrow$ Two extra parameters (θ_4, σ_4) compared to 3-factor model.
$\rightarrow$ One unspanned factor.

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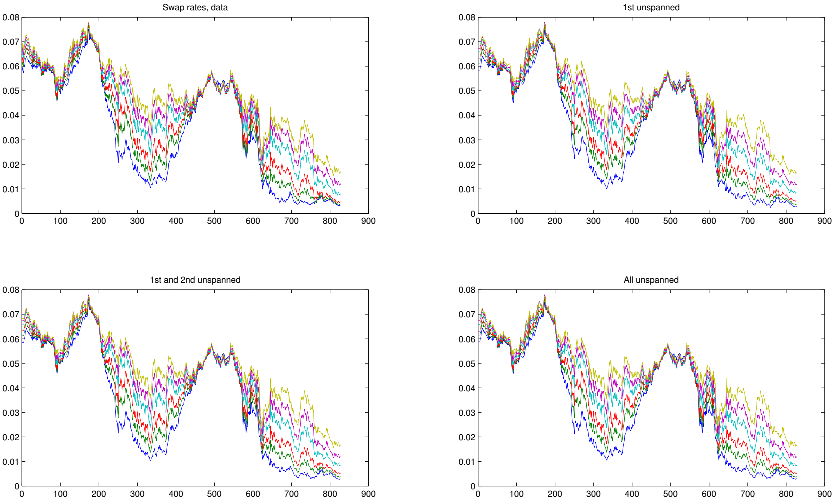
- **Similarly,** we can let first + second or all three factors be unspanned (or other combinations)

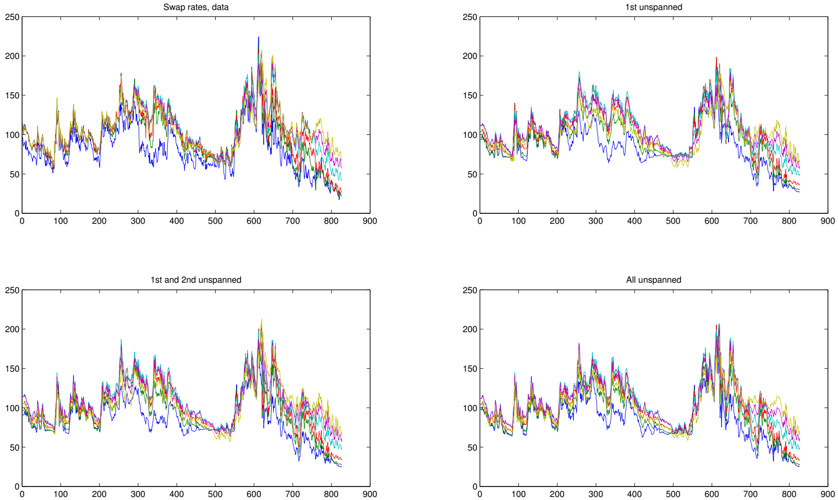
Calibration to swap rates and swaptions

Results for swap rates



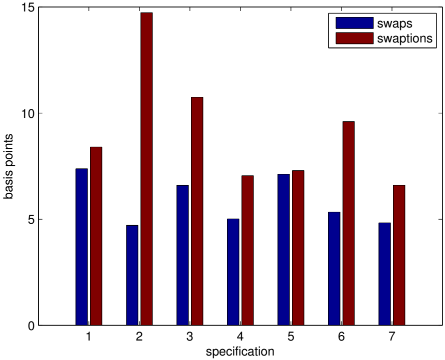
Calibration to swap rates and swaptions

Results for swaption implied volatilities



Calibration to swap rates and swaptions

Camparing USV specifications (Std. dev. of pricing error):



Bars	Factors unspanned	Bars	Factors unspanned
1	1st	5	1st and 3rd
2	2nd	6	2nd and 3rd
3	3rd	7	all three
4	1st and 2nd		

Conclusion

- Processes with affine drift combined with an affine state price density yield a large class of tractable term structure models: The **Linear-Rational term structure models**.
- Unlike affine term structure models, we combine:
  - Explicit bond prices, short rates, forward rates
  - Both risk-neutral and historical dynamics (MPR, risk premie)
  - Nonnegative short rates
  - Simple ways to incorporate USV (crucial for fitting swaptions)
  - Very fast swaption pricing
  - Great fit to market data (swaps + swaptions)