Portfolio Optimisation under Transaction Costs

W. Schachermayer

University of Vienna
Faculty of Mathematics

joint work with
Ch. Czichowsky (Univ. Vienna), J. Muhle-Karbe (ETH Zürich)

We fix a strictly positive càdlàg stock price process $S = (S_t)_{0 \le t \le T}$.

For $0 < \lambda < 1$ we consider the bid-ask spread $[(1-\lambda)S, S]$.

A self-financing trading strategy is a predictable, finite variation process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ such that

$$d\varphi_t^0 \le -S_t (d\varphi_t^1)_+ + (1-\lambda) S_t (d\varphi_t^1)_-$$

φ is called 0-admissible if

$$\varphi_t^0 + (1-\lambda) S_t (\varphi_t^1)_+ - S_t (\varphi_t^1)_- \ge 0$$

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Definition [Jouini-Kallal ('95), Cvitanic-Karatzas ('96), Kabanov-Stricker ('02),...]

A *consistent-price system* is a pair $(\tilde{S}, Q)$ such that $Q \sim \mathbb{P}$, the process $\tilde{S}$ takes its value in $[(1-\lambda)S, S]$, and $\tilde{S}$ is a Q-martingale.

Identifying Q with its density process

$$Z_t^0 = \mathbb{E}\left[\frac{dQ}{d\mathbb{P}} | \mathcal{F}_t\right], \qquad 0 \leq t \leq T$$

we may identify $(\tilde{S}, Q)$ with the $\mathbb{R}^2$-valued martingale $Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T}$ such that

$$\tilde{S} := \frac{Z^1}{Z^0} \in [(1-\lambda)S, S].$$

For $0 < \lambda < 1$, we say that S satisfies (CPS^λ) if there is a consistent price system for transaction costs λ.

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Remark [Guasoni, Rasonyi, S. ('08)]

If the process $S = (S_t)_{0 \leq t \leq T}$ is *continuous* and has *conditional full support*, then (CPS^{μ}) is satisfied, for all $\mu > 0$.
For example, exponential fractional Brownian motion verifies this property.

Portfolio optimisation

The *set of non-negative claims attainable at price x* is

$$\mathcal{C}(x) = \left\{ \begin{array}{l} X_T \in L^0_+ : \textit{there is a } 0\textit{-admissible } \varphi = (\varphi^0_t, \varphi^1_t)_{0 \le t \le T} \\ \textit{starting at } (\varphi^0_0, \varphi^1_0) = (x, 0) \textit{ and ending at} \\ (\varphi^0_T, \varphi^1_T) = (X_T, 0) \end{array} \right\}$$

Given a utility function $U : \mathbb{R}_+ \to \mathbb{R}$ define

$$u(x) = \sup\{\mathbb{E}[U(X_T)] : X_T \in \mathcal{C}(x)\}.$$

Cvitanic-Karatzas ('96), Deelstra-Pham-Touzi ('01), Cvitanic-Wang ('01), Bouchard ('02),...

Question 1

What are conditions ensuring that $\mathcal{C}(x)$ is closed in $L^0_+(\mathbb{P})$. (w.r.t. to convergence in measure) ?

Theorem [Cvitanic-Karatzas ('96), Campi-S. ('06)]:

Suppose that (CPS^μ) is satisfied, for all $\mu > 0$, and fix $\lambda > 0$. Then $\mathcal{C}(x) = \mathcal{C}^\lambda(x)$ is closed in L^0.

Theorem [Guasoni, Rasonyi, S. ('08)]:

Let $S = (S_t)_{0 \le t \le T}$ be a continuous process. TFAE

(i) For each $\mu > 0$, S does not allow for arbitrage under transaction costs μ.

(ii) For each $\mu > 0$, (CPS^μ) holds, i.e. *consistent price systems under transaction costs* μ exist.

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The dual objects

Definition

We denote by $D(y)$ the convex subset of $L^0_+(\mathbb{P})$

$$D(y) = \{yZ^0_T = y\frac{dQ}{d\mathbb{P}}, \textit{for some consistent price system } (\tilde{S}, Q)\}$$

and

$$\mathcal{D}(y) = \overline{sol\ (D(y))}$$

the closure of the solid hull of $D(y)$ taken with respect to convergence in measure.

Definition [Kramkov-S. ('99), Karatzas-Kardaras ('06), Campi-Owen ('11),...]

We call a process $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ a *super-martingale deflator* if $Z_0^0 = 1$, $\frac{Z^1}{Z^0} \in [(1-\lambda)S, S]$, and for each 0-admissible, self-financing φ the value process

$$\varphi_t^0 Z_t^0 + \varphi_t^1 Z_t^1 = Z_t^0 (\varphi_t^0 + \varphi_t^1 \frac{Z_t^1}{Z_t^0})$$

is a super-martingale.

Remark

A consistent price system $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ is a super-martingale deflator.

Proposition

The closure $\mathcal{D}(y)$ of $D(y)$ can be characterized as

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$$\mathcal{D}(y) = \{yZ_T^0 : \; Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T} \textit{ a super} - \textit{martingale deflator}\}$$

Theorem (Czichowsky, Muhle-Karbe, S. ('12))

Let S be a càdlàg process, $0 < \lambda < 1$, suppose that (CPS^{μ}) holds true, for some $0 < \mu < \lambda$, suppose that U has reasonable asymptotic elasticity and $u(x) < U(\infty)$, for $x < \infty$.
Then $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are polar sets:

$$\begin{aligned} X_T \in \mathcal{C}(x) \quad & \text{iff } \langle X_T, Y_T \rangle \leq xy, \quad \text{for } Y_T \in \mathcal{D}(y) \\ Y_T \in \mathcal{D}(y) \quad & \text{iff } \langle X_T, Y_T \rangle \leq xy, \quad \text{for } X_T \in \mathcal{C}(y) \end{aligned}$$

Therefore by the abstract results from [Kramkov-S. ('99)] the duality theory for the portfolio optimisation problem works as nicely as in the frictionless case: for $x > 0$ and $y = u'(x)$ we have

(i) There is a unique primal optimiser $\hat{X}_T(x) = \hat{\varphi}^0_T$ which is the terminal value of a trading strategy $(\hat{\varphi}^0_t, \hat{\varphi}^1_t)_{0 \le t \le T}$.

(i') There is a unique dual optimiser $\hat{Y}_T(y) = \hat{Z}^0_T$ which is the terminal value of a super-martingale deflator $(\hat{Z}^0_t, \hat{Z}^1_t)_{0 \le t \le T}$.

(ii) $U'(\hat{X}_T(x)) = \hat{Z}^0_t(y), \qquad -V'(\hat{Z}_T(y)) = \hat{X}_T(x)$

(iii) The process $(\hat{\varphi}^0_t \hat{Z}^0_t + \hat{\varphi}^1_t \hat{Z}^1_t)_{0 \le t \le T}$ is a martingale, and therefore

$\{d\hat{\varphi}^0_t > 0\} \subseteq \{\frac{\hat{Z}^1_t}{\hat{Z}^0_t} = (1-\lambda)S_t\},$

$\{d\hat{\varphi}^0_t < 0\} \subseteq \{\frac{\hat{Z}^1_t}{\hat{Z}^0_t} = S_t\},$

etc. etc.

Theorem [Cvitanic-Karatzas ('96)]

In the setting of the above theorem *suppose* that $(\hat{Z}_t)_{0\leq t\leq T}$ is a local martingale.
Then $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0} \in [(1-\lambda)S, S]$ is a *shadow price*, i.e. the optimal portfolio for the *frictionless market* $\hat{S}$ and for the *market* S *under transaction costs* λ coincide.

Sketch of Proof

Suppose (w.l.g.) that $(\hat{Z}_t)_{0\leq t\leq T}$ is a true martingale. Then $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \hat{Z}^0_T$ defines a *probability measure* under which the process $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ is a martingale. Hence we may apply the frictionless theory to $(\hat{S}, \mathbb{P})$.
$\hat{Z}^0_T$ is (a fortiori) the dual optimizer for $\hat{S}$.
As $\hat{X}_T$ and $\hat{Z}^0_T$ satisfy the first order condition

$$U'(\hat{X}_T) = \hat{Z}^0_T,$$

$\hat{X}_T$ must be the optimizer for the frictionless market $\hat{S}$ too. ■

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Question

When is the dual optimizer $\hat{Z}$ a *local martingale*?
Are there cases when it only is a *super-martingale*?

Theorem [Czichowsky-S. ('12)]

Suppose that S is *continuous* and satisfies $(NFLVR)$, and suppose that U has reasonable asymptotic elasticity. Fix $0 < \lambda < 1$ and suppose that $u(x) < U(\infty)$, for $x < \infty$.
Then the dual optimizer $\hat{Z}$ is a local martingale. Therefore $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ is a shadow price.

Remark

The condition $(NFLVR)$ cannot be replaced by requiring (CPS^λ), for each $\lambda > 0$.

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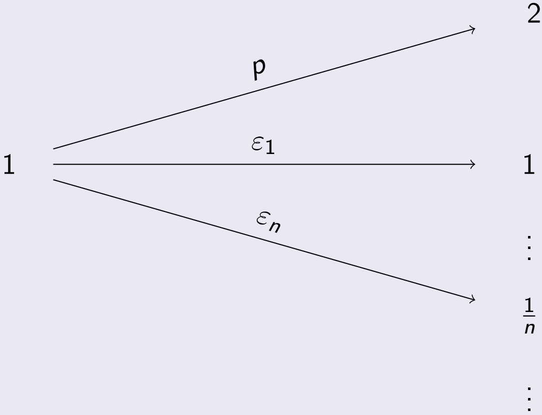
Examples

Frictionless Example [Kramkov-S. ('99)]

Let $U(x) = \log(x)$. The stock price $S = (S_t)_{t=0,1}$ is given by

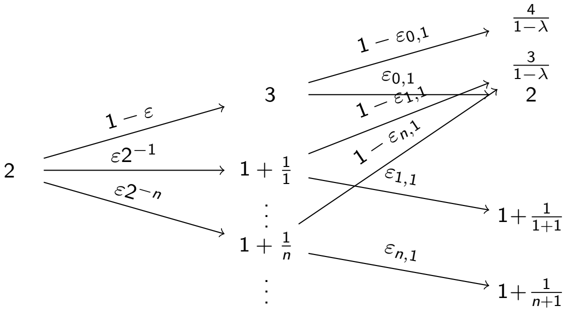


Here $\sum_{n=1}^{\infty} \varepsilon_n = 1 - p \ll 1$.
For $x = 1$ the optimal strategy is to buy one stock at time 0 i.e. $\hat{\varphi}_1^1 = 1$.

Let $A_n = \{S_1 = \frac{1}{n}\}$ and consider $A_\infty = \{S_1 = 0\}$ so that $\mathbb{P}[A_n] = \varepsilon_n > 0$, for $n \in \mathbb{N}$, while $\mathbb{P}[A_\infty] = 0$.

Intuitively speaking, the constraint $\hat{\varphi}_1^1 \leq 1$ comes from the null-set A_∞ rather than from any of the A_n's.
It turns out that the dual optimizer $\hat{Z}$ verifies $\mathbb{E}[\hat{Z}_1] < 1$, i.e. only is a super-martingale. Intuitively speaking, the optimal measure $\hat{Q}$ gives positive mass to the $\mathbb{P}$-null set A_∞ (compare Cvitanic-Schachermayer-Wang ('01), Campi-Owen ('11)).

Discontinuous Example under transaction costs λ (Czichowsky, Muhle-Karbe, S. ('12), compare also Benedetti, Campi, Kallsen, Muhle-Karbe ('11)).



For $x = 1$ it is optimal to buy $\frac{1}{1+\lambda}$ many stocks at time 0. Again, the constraint comes from the $\mathbb{P}$-null set $A_\infty = \{S_1 = 1\}$.

There is no shadow-price. The intuitive reason is again that the binding constraint on the optimal strategy comes from the $\mathbb{P}$-null set $A_\infty = \{S_1 = 1\}$.

Continuous Example under Transaction Costs [Czichowsky-S. ('12)]

Let $(W_t)_{t\geq 0}$ be a Brownian motion, starting at $W_0 = w > 0$, and

$$\tau = \inf\{t : W_t - t \leq 0\}$$

Define the stock price process

$$S_t = e^{t\wedge\tau}, \qquad t \geq 0.$$

S does not satisfy $(NFLVR)$, but it does satisfy (CPS^λ), for all $\lambda > 0$.
Fix $U(x) = \log(x)$, transaction costs $0 < \lambda < 1$, and the initial endowment $(\varphi_0^0, \varphi_0^1) = (1, 0)$.
For the trade at time $t = 0$, we find three regimes determined by thresholds $0 < \underline{w} < \bar{w} < \infty$.

(i) if $w \leq \underline{w}$ we have $(\hat{\varphi}^0_{0_+}, \hat{\varphi}^1_{0_+}) = (1, 0)$, i.e. no trade.
(ii) if $\underline{w} < w < \bar{w}$ we have $(\hat{\varphi}^0_{0_+}, \hat{\varphi}^1_{0_+}) = (1 - a, a)$, for some $0 < a < \frac{1}{\lambda}$.
(iii) if $w \geq \bar{w}$, we have $(\hat{\varphi}^0_{0_+}, \hat{\varphi}^1_{0_+}) = (1 - \frac{1}{\lambda}, \frac{1}{\lambda})$, so that the liquidation value is zero (maximal leverage).

We now choose $W_0 = w$ with $w > \bar{w}$.
Note that the optimal strategy $\hat{\varphi}$ *continues to increase the position in stock*, as long as $W_t - t \geq \bar{w}$.

If there were a shadow price $\hat{S}$, we therefore necessarily would have

$$\hat{S}_t = e^t, \qquad \textit{for } 0 \leq t \leq \inf\{u : W_u - u \leq \bar{w}\}.$$

But this is absurd, as $\hat{S}$ clearly does not allow for an e.m.m.

Problem

Let $(B_t^H)_{0 \leq t \leq T}$ be a fractional Brownian motion with Hurst index $H \in \ ]0,1[\setminus \{\frac{1}{2}\}$. Let $S = \exp(B_t^H)$, and fix $\lambda > 0$ and $U(x) = \log(x)$.

Is the dual optimiser a local martingale or only a super-martingale?

Equivalently, is there a shadow price $\hat{S}$?